REMARKS ON A PAPER OF HOBBY AND WRIGHT

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C. Hobby and C. R. B. Wright [2] have just published the following Theorem A. However, their proof seems to contain an error.²

The notation of [2] is used except that $G_n$ is not reserved for the $n$th term of the lower central series of $G$; $\phi(G)$ denotes the Frattini subgroup of $G$; $(G, H)$ means the group generated by the commutators $g^{-1}h^{-1}gh$ where $g \in G$, $h \in H$; $(A_1, A_2, \cdots, A_{n+1})$ is defined inductively as $((A_1, A_2, \cdots, A_n), A_{n+1})$; $H \leq G$ means that $H$ is properly included in $G$.

**Theorem A.** If $G$ is a finite $p$-group and $H$ a subgroup of $G$ such that $H \trianglelefteq G_n$, then $(H \phi(G_n)) \trianglelefteq G_n$, where $X_n$ denotes the $n$th term of the lower central series of $X$.

N. Itô [3] had already proved this theorem for the case $n = 2$. In this note, Itô's theorem is generalized in a somewhat stronger form than Theorem A. In fact, as was shown in [2], if Theorem A were false, it would have to fail for a normal subgroup $H$ of $G$. In the presence of this fact, Theorem A is contained in

**Theorem B.** Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$ be a nondecreasing finite chain of normal subgroups of a finite $p$-group $G$ and let $H_1, H_2, \cdots, H_n$ be normal subgroups of $G$ with $H_i \trianglelefteq G_i$ for all $i$. If

$$(H_1, H_2, \cdots, H_n) \subseteq (G_1, G_2, \cdots, G_n),$$

then

$$(H_1\phi(G_1), H_2\phi(G_2), \cdots, H_n\phi(G_n)) \subseteq (G_1, G_2, \cdots, G_n).$$

**Proof.** Suppose that the theorem is false for a certain $n$. Let $G$ be of minimal order for which it is false and let $H_1, H_2, \cdots, H_n$ be chosen such that if $K_i$ is a normal subgroup of $G$ and $H_i \subseteq K_i \subseteq G_i$ for any $i$, then

$$(H_1, H_2, \cdots, K_i, \cdots, H_n) = (G_1, G_2, \cdots, G_n).$$

For convenience, set $(H_1, H_2, \cdots, H_n) = A$, $(H_1\phi(G_1), H_2\phi(G_2), \cdots, H_n\phi(G_n)) = B$, and $(G_1, G_2, \cdots, G_n) = C$. First, it is noted that


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2 The fact that $G$ is noncyclic does not imply that $\phi(G)$ is the intersection of all normal subgroups of index $p^k$ in $G$.
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Thus if \( A \neq \langle 1 \rangle \), the relation

\[
B/A = (H_1\phi(G_1)/A, \cdots, H_n\phi(G_n)/A) \subset C/A
\]

holds according to the choice of \( G \) since in general \( \phi(G/N) = N\phi(G)/N \).
However, this implies that \( B \subset C \); hence \( A = \langle 1 \rangle \).

Let \( z \) be an element of order \( p \) in \( Z \), the center of \( G \). If the relation

\[
\langle z \rangle /\langle z \rangle = ((z)H_1/\langle z \rangle, \cdots, (z)H_n/\langle z \rangle)
\subset ((z)G_1/\langle z \rangle, \cdots, (z)G_n-1/\langle z \rangle, (z)G_n/\langle z \rangle) = \langle z \rangle C/\langle z \rangle
\]

holds, then

\[
\langle z \rangle B/\langle z \rangle = ((z)H_1\phi(G_1)/\langle z \rangle, \cdots, (z)H_n\phi(G_n)/\langle z \rangle) \subset \langle z \rangle C/\langle z \rangle,
\]

which implies that \( B \subset C \). Thus \( C = \langle z \rangle \).

Since all the subgroups involved are normal, from the linearity properties of commutators [1, p. 150] it follows that

\[
(H_1, \cdots, H_{k-1}, H_k\phi(G_k), H_{k+1}, \cdots, H_n)
\]

\[
= A(H_1, \cdots, H_{k-1}, \phi(G_k), H_{k+1}, \cdots, H_n) = \langle 1 \rangle.
\]

But \( H_k \subset H_k\phi(G_k) \) for some \( k \). Thus a contradiction has been established on the choice of the \( H_i \), and the theorem is proved.

References


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