A SIMPLE PROOF OF FROBENIUS’S INTEGRATION THEOREM

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The connection between Stokes’s Integral Theorem and the Frobenius-Cartan Integration Theorem concerning Pfaffian systems has been noted a long time. In this note, we generalize Stokes’s theorem to implicit vector valued differential forms and derive from it a general Frobenius theorem concerning mappings in Banach spaces. The only difficulty in the proof arises in the need to show differentiability with respect to a parameter of solutions of a certain differential equation, but is is easily overcome. The generality of the theorem seems to be necessary for applications to the new subjects of infinite groups and of differential geometry in infinitely many dimensions. E.g., it allows us to associate a local group to any infinite-dimensional Lie algebra in a Banach space. For finite dimensional vector spaces we obtain the classical theorem with nearly minimal differentiability conditions [4]. Also for finite dimensional spaces, one might derive from it parts of the Cartan-Kähler theory of integral manifolds [3] for not completely integrable $C^\infty$ systems.

1. All spaces in this note are real or complex Banach spaces. The only topology to be considered is the norm topology and the topologies induced by it in the spaces of linear mappings. A mapping will always be a bounded linear transformation of a Banach space into another one, a function is a continuous map of spaces. Given two spaces $E, F$ with neighborhoods $U \subset E, V \subset F$, a differential form is a function $A(x, y): U \times V \to \mathcal{L}(E, F)$, taking values in the space of all mappings of $E$ into $F$. We will denote by $k = A(x, y)h, h \in E, k \in F$ the image of $h$ under the mapping, image of $(x, y)$.

A function $f(x): U \to F$ is said to be (Fréchet) differentiable in $x_0 \in U$, if $df(x_0)h = \lim_{h \to 0} (1/\lambda)(f(x_0 + \lambda h) - f(x_0))$ defines a mapping of $E$ into $F$, and if furthermore there exists for every neighborhood $V$ of the zero of $F$ an $\varepsilon(V) > 0$ such that

$$f(x_0 + h) - f(x_0) - df(x_0)h \in \|h\|V$$

for all $h$ satisfying $0 < \|h\| < \varepsilon(V)$, $x_0 + h \in U$. Frobenius’s problem may then be stated as follows: Given a differential form $A(x, y): U \times V \to \mathcal{L}(E, F)$, find a function $f(x): U \to F$ such that $df(x) = A(x, f(x))$, under the initial condition $f(x_0) = y_0, x_0 \in U, y_0 \in V$.

The Fréchet differential is a straightforward generalization of the
directional derivative. For finite dimensional spaces, $A(x, y)$ is a vector valued differential form, or a system $(\omega^i)$ of Pfaffian forms. Our problem then is to find a vector of functions $f^i(x', \cdots, x^n)$ such that $df^i - \omega^i = 0$ under given initial conditions and that the Jacobian $\partial(f^i)/\partial(x^j)$ be of maximum rank.

A differential form may itself be differentiable. Its differential may be made explicit by the partial differentials $d_x$ and $d_y$, operating on $E$ and $F$ respectively:

$$[dA(x, y)h]k \times l = [d_xA(x, y)h]k + [d_yA(x, y)h]l, \ h, k \in E, l \in F.$$  

If $y$ is given by a Pfaffian equation $dy = A(x, y)$, the substitution $l = dy(x)k = A(x, y)k$ induces from $dA$ a function $U \times V \rightarrow \mathcal{E}(E \wedge E, F)$, the formal exterior differential (in E. Cartan's terminology: the exterior differential mod $dy = A(x, y) \wedge$):

$$\delta A(x, y)(h \wedge k) = [d_xA(x, y)h]k + [d_yA(x, y)h]A(x, y)k - [d_xA(x, y)k]h - [d_yA(x, y)k]A(x, y)h.$$  

Frobenius's problem is then solved by the

**Theorem.** Assume $dA(x, y)$ to be a bounded continuous function $U \times V \rightarrow \mathcal{E}(E, \mathcal{E}(E, F))$. The equation

$$dy = A(x, y)$$

subject to the initial condition

$$y(x_0) = y_0, \quad x_0 \in U, \ y_0 \in V$$

has a unique continuously differentiable solution in some neighborhood of $x_0$ if and only if $\delta A(x, y) = 0$ in $U \times V$.

For functions $f: E \rightarrow F$ we may write $df = \delta f$, since $E = \otimes^1E = \wedge^1E$. Under our conditions a symmetry property holds for second differentials which implies $\delta \delta f = 0$. This takes care of the necessity part of the theorem, if it is assumed that a solution exists for arbitrary initial data.

2. A curve is a function $c: I \rightarrow E$ defined on the unit interval $I = [0, 1]$. We will denote by $c_t$ its restriction to $I_t = [0, t]$, $0 \leq t \leq 1$. All curves considered in the sequel are $C^1$, hence rectifiable. Given a differential form $B(x): U \rightarrow \mathcal{E}(E, F)$ and a curve $c: I \rightarrow U$, $U \subseteq E$, the integral of $B$ on $c_t$ is the integral on $I_t$

$$\int_{c_t} B(x)dx = \int_0^t B(c(\tau))c'(\tau)d\tau.$$
Here and in the future we will restrict ourselves to convex (e.g., spherical) neighborhoods. Any two points \( x_0, x_1 \) in \( U \) may then be joined by a \( C^1 \) curve in \( U \), e.g., the segment \( c(t) = (1-t)x_0 + tx_1 \). In order to solve (1) we first join \( x_0 \) to a nearby \( x_1 \) by a smooth curve \( c \) and solve the integral equation

\[
y(t) = y_0 + \int_{c(t)} A(x, y(t))dx
\]

\[
= y_0 + \int_0^t A(c(\tau), y(\tau))c'(\tau)d\tau.
\]

By our assumption the Fréchet differential \( d_A(x, y) \) is a bounded linear operator, hence \( A(x, y) \) satisfies a Lipschitz condition in \( y \). The usual successive approximations \([1]\) assure us that (3) has a unique solution for \( x_1 \) in some convex neighborhood \( U(x_0) \), and that \( dy(t)dx = A(c(t), y(t))dx \) holds for \( dx = c'(t)dt \) and all \( t \in I \).

In order to prove our theorem, we will join \( x_0 \) and \( x_1 \) by any other smooth curve \( c^* \) in \( U \). Let \( y^*(t) \) be the solution of (3) corresponding to \( c^* \). Under the conditions of our theorem, we show that \( y(1) = y^*(1) \), i.e. \( y(x) \) is uniquely defined by integration along smooth curves. Finally, we have to show that (1) holds for this \( y(x) \) for all \( h \in E \). In view of the last sentence of the preceding paragraph this will be established if for any \( h \in E \), \( ||h|| < \epsilon \), we find a \( c \) with \( c(0) = x_0, c(1) = x_1, c'(1) = h \). Take in \( U \) a spherical neighborhood about \( x_1 \), with radius \( \epsilon/2 \). Then for \( ||h|| < \epsilon \) the point \( x_1 - h/2 \) is in that neighborhood, hence the “parabola” \( c(t) = (1-t^2)x_0 + 2t(1-t)(x_1 - h/2) + t^2x_1 = (1-t)\{ (1-t)x_0 + t(x_1 - h/2) \} + t(1-t)(x_1 - h/2) + tx_1 \) is in \( U \); this is a curve of the desired property.

3. Let \( c \) and \( c^* \) be two smooth curves joining \( x_0 \) and \( x_1 \) in \( U \). A homotopy of \( c \) and \( c^* \) is defined by

\[
c_s(t) = (1-s)c(t) + sc^*(t), \quad s \in I.
\]

All \( c_s(t) \) are \( C^1 \) curves, for fixed \( s \). By hypothesis, \( \sup_{x \in U, y \in V} ||A(x, y)|| < \infty \), hence (3) has a solution for any \( c_s \) and some \( \delta, ||x_1 - x_0|| < \delta \) (cf. \([1, \text{Theorem 1}]\)). As a preliminary step, we have to study the differentiability with respect to \( s \) of this \( y_s(t) \), given by

\[
y_s(t) = y_0 + \int_0^t A(c_s(\tau), y_s(\tau))((1-s)c'(\tau) + sc'^*(\tau))d\tau.
\]

If \( dy_s(t)/ds \) exists and is continuous on the compact set \( I \times I \), the function \( s_*(t) = dy_s(t)/ds \) satisfies the integral equation
\[ z_\circ(t) = \int_0^t \left[ d_A(a(t), y_\circ(t))((1 - s)c'(r) + sc^{*\prime}(r)) \right] (c^*(r) - c(r)) \, dr \]

\[ + \int_0^t A(a(t), y_\circ(t))(c^{*\prime}(r) - c'(r)) \, dr \]

\[ + \int_0^t \left[ d_A(a(t), y_\circ(t))((1 - s)c'(r) + sc^{*\prime}(r)) \right] z_\circ(r) \, dr, \]

which is of the type

\[ (6a) \quad z_\circ(t) = P(s, t) + \int_0^t Q(s, r) z_\circ(r) \, dr \]

with continuous (hence bounded) \( P(s, t) \) and \( Q(s, t) \) \((s, t) \in I \times I \). Both (5) and (6) have unique solutions which are continuous on the whole of \( I \times I \) (use [1, Theorem 3]), hence we may represent \( y_\circ(t) \) as a Stieltjes integral in the variable \( s \); using the abbreviation introduced in (6a) we have

\[ y_\circ(t) - y_\circ(0) = \int_0^t d_\circ y_\circ(t) = \int_0^t P(s, t) \, ds + \int_0^t \int_0^s Q(s, r) d_\circ y_\circ(r) \, dr. \]

Introducing \( Z(s, t) = \int_0^t d_\circ y_\circ(t) \, ds + y_\circ(t) - y_\circ(s) \), we finally have an integro-differential equation

\[ (7) \quad Z(s, t) = \int_0^t \int_0^s Q(s, r) d_\circ Z(s, r) \, dr, \quad Z(0, t) = 0. \]

We say that \( Z(s, t) \) is Lipschitzian in \( s \). For the integral this is trivial; to see it for \( y_\circ(t) \) remark that \( A(a(t), y_\circ(t)) \) is uniformly continuous on the compact set \((s, t) \in I \times I \). If we denote by \( L(c) \) the length of the curve \( c \), we see from (3) that

\[ ||y_\circ(t) - y_\circ(0)|| < (||A|| + \varepsilon)(L(c_\circ) + L(c_\circ)) \cdot |s - s'| \]

if for \( \varepsilon > 0 \) we choose \( \delta \) to have \( |s - s'| < \delta \) imply \( ||A(c_\circ(t), y_\circ(t)) - A(c_\circ(t), y_\circ(t))|| < \varepsilon \). We know now that there exists a constant \( Z \),

\[ ||Z(s, t) - Z(s, t)|| < Z |s - s'| \quad \text{for} \quad |s - s'| < \delta. \]

This is sufficient to use the standard procedure to infer that \( Z(s, t) = 0 \) is the only solution of (7), as we have by successive approximations \( ||Z(s, t)|| < ||Q||Zst \), hence \( ||Z(s, t)|| < 2^{-k}||Q||Zst^k/k! \) for all natural \( k \). Therefore \( y_\circ(t) \) is a \( C^1 \) function in \( s \).

4. By partial integration of the second term in (6) we obtain
\[
\frac{dy_1(t)}{ds} = \int_0^1 \left[ d_x A(x(t), y(t)) \left\{ (1-s)c'(t) + sc^*(t) \right\} \left( c^*(t) - c(t) \right) \right] \, dt
\]

\[
- \int_0^1 \left[ d_x A(x(t), y(t)) \left( c^*(t) - c(t) \right) \right] \left\{ (1-s)c'(t) + sc^*(t) \right\} \, dt
\]

\[
+ \int_0^1 \left[ d_y A(x(t), y(t)) \left( c^*(t) - c(t) \right) \right] \frac{\partial y_1(t)}{\partial t} \, dt
\]

As \( dx = \{ (1-s)c'(t) + sc^*(t) \} \, dt + \{ c^*(t) - c(t) \} \, ds, \ dy = (\partial y_1(t)/\partial t) \, dt \)

\[
+ (\partial y_1(t)/\partial s) \, ds, \text{ we finally have our Stokes's formula}
\]

\[
g_1(1) - g_0(1) = \int_0^1 \frac{dy_1(t)}{ds} \, ds
\]

\[
= \int \int_{x \in \Omega} \left\{ d_x A(x, y) dx \wedge dy \right\} + d_y A(x, y) dx \wedge dy
\]

\[
= \int \int_{x \in \Omega} \delta A(x, y) dx,
\]

which from (5) may be written in an easily understood shorthand

\[
(8) \quad \oint_{c \in \gamma} A(x, y) dx = \int \int_{x \in \Omega} \delta A(x, y) dx
\]

and from which by the assumption of our theorem we have

\[
g_1(1) = g_0(1),
\]

or, in our previous notation \( y(x_1) = y^*(x_1). \) This completes the proof.

In the finite dimensional case,

\[
\dim y(U) = \min(\dim U, \min_{x \in \partial U} \dim A(x, F)).
\]

**Bibliography**


