A CHARACTERIZATION OF MONOMIALS

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A monomial of \( n \) complex variables is a function of the form

\[
P(z_1, \ldots, z_n) = \sum_{\gamma \in \mathbb{N}^n} a_\gamma z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}
\]

where \( p_1, \ldots, p_n \) are non-negative integers and where \( a \) is constant. The set

\[
E = \{(z_1, \ldots, z_n) \mid |z_1| < 1, \ldots, |z_n| < 1\}
\]

is called the unit polycylinder. The set

\[
D = \{(z_1, \ldots, z_n) \mid |z_1| = 1, \ldots, |z_n| = 1\}
\]

is said to be the distinguished boundary of \( E \). Note that \( D \) is not the whole boundary of \( E \).

We want to prove the following theorem:

The monomials are the only entire functions whose absolute value is constant on the distinguished boundary of the unit polycylinder.

Proof. Denote by \( \mathbb{C}^n \) the space of \( n \) complex variables. The elements are the vectors

\[
z = (z_1, \ldots, z_n)
\]

whose coordinates \( z_i \) are complex numbers.

Now, let \( f \) be an entire function, whose absolute value \( |f| \) is constant on \( D \). If \( |f| \) is identically zero on \( D \), then \( f \) is identically zero on \( \mathbb{C}^n \). Therefore, \( f \) is a monomial.

We may exclude this case and assume, without loss of generality, that \( |f(z)| = 1 \) for \( z \in D \). Now, we want to show that such an entire function \( f \) is either constant or has zeros in \( E \). Assume \( f(z) \neq 0 \) for \( z \in E \). Then, we have by a well-known theorem

\[
\begin{align*}
\max_{z \in \overline{E}} |f(z)| &= \max_{z \in \overline{D}} |f(z)| = 1, \\
\min_{z \in \overline{E}} |f(z)| &= \min_{z \in \overline{D}} |f(z)| = 1.
\end{align*}
\]

Therefore, \( f \) is constant. Consequently, \( f \) is either constant or has zeros in \( E \).

Now, we want to prove that \( f(z) \neq 0 \) for all \( z \) in

\[
A = \{(z_1, \ldots, z_n) \mid z_1 z_2 \cdots z_n \neq 0\}.
\]

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Define the function \( g \) by
\[
g(z_1, \ldots, z_n) = f\left(\frac{1}{\bar{z}_1}, \ldots, \frac{1}{\bar{z}_n}\right)
\]
on \( A \). This function is holomorphic on \( A \) since its real partial derivatives exist, are continuous, and satisfy the Cauchy-Riemann equations
\[
g_z^*(z_1, \ldots, z_n) = \frac{\partial}{\partial z_j} f z_j = \frac{1}{\bar{z}_j}.
\]
For \( z \in D \subset A \), we have
\[
g(z_1, \ldots, z_n) = f\left(\frac{1}{\bar{z}_1}, \ldots, \frac{1}{\bar{z}_n}\right) = f(z_1, \ldots, z_n)
\]
Therefore
\[
g(z_1, \ldots, z_n) \cdot f(z_1, \ldots, z_n) = 1 \quad \text{for } (z_1, \ldots, z_n) \in D.
\]
The function \( g \cdot f \) is holomorphic on \( A \) and identically one on \( D \). Since \( A \) is a connected, open neighborhood of \( D \), the function \( g \cdot f \) is equal to one on \( A \):
\[
f(z) \cdot g(z) = 1 \quad \text{for } z \in A.
\]
Therefore, we have \( f(z) \neq 0 \) for \( z \in A \).

Our function \( f \) vanishes at most on the planes \( \{(z_1, \ldots, z_n) | z_r = 0\} \). Therefore \( f \) has the form \( ^1 \)
\[
f(z_1, \ldots, z_n) = z_1^{p_1} \cdots z_n^{p_n} \cdot h(z_1, \ldots, z_n)
\]
where \( p_1, \ldots, p_n \) are non-negative integers and \( h \) is an entire function which does not vanish at all, and whose absolute value is constant on \( D \). Consequently, \( h \) is a constant. We obtain
\[
f(z_1, \ldots, z_n) = a z_1^{p_1} \cdots z_n^{p_n}
\]
q.e.d.

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