FINITE COLLECTIONS OF 2-SPHERES IN E^3

B. J. BALL

If G is a finite collection of simple closed curves in E^2, E is a closed subset of the sum of their interiors which does not intersect any of them, and ϵ is a positive number, then there exists a finite collection H of mutually exclusive simple closed curves with mutually exclusive interiors such that E is a subset of the sum of the interiors of the members of H and each member of H lies in the ϵ-neighborhood of the sum of the members of G. This theorem can be easily proved by observing, first, that it may be assumed without loss that no member of G is contained in the interior of any other member of G, and then that this condition implies that each component of E^2-G^*, where G^* denotes the sum of the members of G, is bounded by a simple closed curve.

The analogous theorem in E^3 (with the elements of G being 2-spheres) does hold, but is not susceptible of such a simple proof, mainly because of the increased variety possible in the components of E^3-G^*. Nevertheless, it would seem to be a useful theorem in the study of E^3, and the present paper is devoted to a proof of this proposition (Theorem 2). Certain special cases of this theorem have been treated by Harrold ([**4**, pp. 618–619] and implicitly in [**3**]). In particular, if E is assumed to have the property that each of its components can be enclosed by arbitrarily small 2-spheres not intersecting E, then the conclusion of Theorem 2 can be shown to follow from the results of [**3**].

In view of the approximation theorem of Bing, only the case in which the members of G are polyhedral need be considered.

THEOREM 1. *Suppose P_1 is the union of a finite number of disjoint polyhedral 2-spheres in E^3, P_2 is a polyhedral 2-sphere in E^3 and E is a closed subset of the union of the interior of P_2 and the interiors of the 2-spheres lying in P_1, such that E does not intersect $P_1\cup P_2$. Then if ϵ is a positive number, there exists a finite collection H of disjoint polyhedral 2-spheres with disjoint interiors such that* (1) *E is a subset of the sum of the interiors of the members of H and* (2) *each member of H lies in the ϵ-neighborhood of $P_1\cup P_2$.*

If P_2 is contained in the interior of one of the 2-spheres lying in P_1, then the collection of all 2-spheres in P_1 which do not enclose other

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2-spheres of P_1 can be used for H. Otherwise, let f_1 be a piecewise linear homeomorphism of E^3 onto itself such that $f_1(P_1)$ is the union of disjoint geometric cubes (the existence of such a homeomorphism follows easily from Theorem 1 of [5]); since the unbounded component of $E^3-f_1(P_1)$ contains a point of $f_1(P_2)$, there exists a polygonal topological ray r having its end point, but no other point, on $f_1(P_2)$ and not intersecting $f_1(P_1)$. There exists a piecewise linear homeomorphism f_2 of E^3 onto itself such that $f_2(r)$ is a straight ray r' and such that there exists a 2-simplex σ on $f_2f_1(P_2)$ such that if $p\in\sigma$ and r_p is the ray parallel to r' with endpoint p, then r_p has only the point p in common with $f_2f_1(P_2)$ and does not intersect $f_2f_1(P_1)$. Let $Y=\bigcup\{r_p \mid p\in \operatorname{Bd}\sigma\}$ and let X denote the closure of $f_2f_1(P_2)-\sigma$. Then $X\cup Y$ is a polygonal set which is homeomorphic to a plane and which separates $f_2f_1(P_1)-(f_2f_1(P_2)\cup \operatorname{Int} f_2f_1(P_2))$ from $\operatorname{Int} f_2f_1(P_2)$ in E^3. There is a piecewise linear homeomorphism f_3 of E^3 onto itself which takes $X\cup Y$ onto a geometric cube C_2 whose upper face is $f_3(X)$. Let $f=f_3f_2f_1$. Then f is a piecewise linear homeomorphism of E^3 onto itself which takes P_2 onto C_2, and $f(E)-\operatorname{Int} C_2$ lies entirely above the xy-plane.

Let $C_1=f(P_1)$. Since P_1 is the union of a finite number of disjoint polyhedral 2-spheres, it is a polyhedron; since f is piecewise linear, C_1 is also a polyhedron. A slight deformation of C_1 takes it onto a polyhedron which has no vertex on the xy-plane; for convenience, it will be assumed that C_1 itself has this property. There is a positive number δ such that $f^{-1}(V_\delta(C_1\cup C_2))\subset V_\epsilon(P_1\cup P_2)$. ($V_r(M)$ denotes the r-neighborhood of the point set M.) Consequently, if it can be shown that there exists a collection H' of disjoint polyhedral 2-spheres with disjoint interiors satisfying conditions (1) and (2) above with E replaced by $f(E)$ and ϵ replaced by δ, then the theorem follows.

For each real number t, let α_t denote the plane $z=t$ and let U_t denote the set of all points of E^3 lying above α_t. Let I denote the sum of the interiors of the 2-spheres contained in C_1 and let $F=f(E)$.

LEMMA 1. *If I' is a component of $I\cap U_0$, then there exists a connected polyhedral 2-manifold M such that* (1) *I' is the interior of M*, (2) *each of $M\cap C_1$ and $M\cap\alpha_0$ is a 2-manifold with boundary*, (3) $(M\cap C_1)\cup(M\cap\alpha_0)=M$ *and* (4) $\operatorname{Bd}(M\cap C_1)=\operatorname{Bd}(M\cap\alpha_0)$.

PROOF. Let $K=C_1\cap\bar{I}'$. If p is a point of $K\cap U_0$, there is a neighborhood V of p in C_1 such that V is an open 2-cell and does not intersect α_0. Hence $K\cap U_0$ is an open 2-manifold.

Now suppose $p\in K\cap\alpha_0$. Let T be a triangulation of E^3 under which

C_1 is polyhedral and has no vertex on α_0. Then either p is in a 2-cell of T or p is in a 1-cell of T.

Suppose p is in a 2-cell σ of T. Let $V=\sigma\cap\overline{U}_0$; then V is a neighborhood of p in K and $\overline{V}$ is a 2-cell.

Suppose p is in a 1-cell τ of T. Let σ_1 and σ_2 be the two 2-cells of T which have τ as an edge. Let $V=(\sigma_1\cap\overline{U}_0)\cup(\sigma_2\cap\overline{U}_0)\cup(\tau\cap\overline{U}_0)$. Then V is a neighborhood of p in K and $\overline{V}$ is a 2-cell.

Hence K is a 2-manifold with boundary, and $\mathrm{Bd}(K)=K\cap\alpha_0$. Let $L=\mathrm{Bd}(K)$. Since no vertex of C_1 is on α_0, it follows that every component of L is a simple closed curve. Let $K'=\alpha_0\cap\overline{I}'$. Then $L\subset K'$. If $p\in K'-L$, there is an open 2-cell V containing p and lying in $K'-L$, so $K'-L$ is an open 2-manifold. Now suppose $p\in L$. Let J be the component of L containing p. Either Int J or Ext J, but not both, is "locally contained" in K', so it follows that there is a neighborhood V of p in K' such that $\overline{V}$ is a closed 2-cell. Hence K' is a 2-manifold with boundary and Bd $K'=L$.

Let $M=K\cup K'$. From the above remarks, it follows that if $L=K\cap K'$ and p is a point of L, there is an arc rps of L and two arcs rp_1s and rp_2s such that $(rp_is)\cap L=\{r\}\cup\{s\}$, rp_1s lies except for its endpoints wholly in $K-L$, rp_2s lies except for its endpoints wholly in $K'-L$, $rp_1s\cup rps$ is the boundary of a neighborhood V of p on K and $rp_2s\cup rps$ is the boundary of a neighborhood V' of p on K'. Then $V\cup V'$ is a neighborhood of p on $K\cup K'$ and is bounded by $rp_1s\cup rp_2s$. Hence M is a 2-manifold without boundary. Furthermore, since $\overline{I}'-I'=M$, it follows that M is connected and $I'=\mathrm{Int}\ M$. (Suppose M is the union of two disjoint closed sets A and B. Let U_A and U_B be disjoint open sets containing A and B, respectively. Then $I'-(U_A\cup U_B)$ is a nonempty, bounded open subset of E^3 and consequently is not closed. But any limit point of $I'-(U_A\cup U_B)$ which does not belong to it is a point of $\overline{I}'-I'$. Since $\overline{I}'-I'=M\subset U_A\cup U_B$, this is a contradiction. Hence M is connected. Since I' is connected, it is a subset either of Int M or of Ext M. Let r be a topological ray (unbounded) starting from a point p of M and having only p in common with M. If $I'\subset\mathrm{Ext}\ M$, then $I'\cap[r-\{p\}]\neq\emptyset$, so $r-\{p\}\subset I'$. But this is impossible since I' is bounded. Hence $I'\subset\mathrm{Int}\ M$. Now if $I'\neq\mathrm{Int}\ M$, there is a point of Int M which is a limit point of I' but does not belong to it. This is impossible since $\overline{I}'-I'=M$.) Thus M satisfies all the conditions of the lemma.

The proof of Lemma 2 below was supplied to the author in a series of conversations with Professor T. R. Brahana.

LEMMA 2. *Suppose X is a compact, connected, orientable 2-manifold* *and* X_i $(i=1, 2)$ *is a* 2*-manifold with boundary which is homeomorphic to a subset of a plane. Suppose furthermore that*

(1) $X=X_1\cup X_2$.

(2) $X_1\cap X_2=\text{Bd } X_1=\text{Bd } X_2$.

(3) *Every* 1*-cycle in* X_1 *is homologous to zero in* X.

Then X *is a* 2*-sphere.*

PROOF. Let m_i $(i=1, 2)$ be the inclusion map of X_i into X. Because of condition (3), $m_{1*}: H_1(X_1)\to H_1(X)$ is the zero map. Since X_2 is homeomorphic to a subset of a plane, each component of X_2 is a disk with a finite number of holes. It follows that every 1-cycle in X_2 is homologous to a linear combination of boundary curves of X_2, and hence since Bd $X_2\subset X_1$, every 1-cycle in X_2 is homologous to zero in X. Thus the map $m_{2*}: H_1(X_2)\to H_1(X)$ is the zero map.

The triad $(X; X_1, X_2)$ is *proper* in the sense of Eilenberg-Steenrod [**2**, p. 34] and hence the following sequence is exact [**2**, p. 39, Theorem 15.3]:

$$
\begin{aligned}
&0 \xleftarrow{\Delta_0} H_0(X) \xleftarrow{\phi_0} H_0(X_1) + H_0(X_2) \xleftarrow{\psi_0} H_0(A) \xleftarrow{\Delta_1} H_1(H) \\
&\xleftarrow{\phi_1} H_1(X_1) + H_1(X_2) \xleftarrow{\psi_1} H_1(A) \xleftarrow{\Delta_2} H_2(X) \xleftarrow{\phi_2} 0.
\end{aligned} \tag{1}
$$

Here $A=X_1\cap X_2$ and Δ_q, ϕ_q and ψ_q are homomorphisms whose definitions are irrelevant except for ϕ_1, which is defined by the equation

$$\phi_1(v_1, v_2) = m_{1*}(v_1) + m_{2*}(v_2), \qquad \text{for } (v_1, v_2) \in H_1(X_1) + H_1(X_2).$$

Since m_{1*} and m_{2*} are zero maps, it follows that ϕ_1 is the zero map.

The following well-known and easily proved proposition will be needed:

If V_1 *and* V_2 *are finite dimensional vector spaces and* f *is a mapping of* V_1 *into* V_2, *then* $\dim f(V_1)=\dim V_1-\dim f^{-1}(0)$.

If mod 2 homology is used, the groups in the sequence (1) will be vector spaces over the field of integers mod 2, so the preceding proposition will apply. Because of the exactness of the sequence, it follows that if f is any one of the maps ϕ_q, Δ_q, ψ_q and g is the next map to the left in (1), then

(2) *the sum of the rank of the images of* g *and the rank of the image of* f *is the rank of the group on which* f *is defined.*

Let n denote the number of components of A and let n_i $(i=1, 2)$ denote the number of components of X_i. Then $H_0(A)$ and $H_1(A)$ each have rank n, $H_0(X_i)$ has rank n_i and $H_1(X_i)$ has rank $n-n_i$. Let k denote the rank of $H_1(X)$. In the following table, each number in the second row represents the rank of the group directly above it, and the number in the third row represents the rank of the image of the incoming homomorphism. From (2), it follows that the sum of any two adjacent numbers in the third row is equal to the second-row entry immediately above the rightmost of these two. In the third row, the zeros at the ends are obtained directly from the homomorphisms involved, and the other numbers are obtained by repeated application of the above rule.

$$
\begin{array}{c|c|c|c|c|}
0 \leftarrow & H_0(X) \leftarrow & H_0(X_1) + H_0(X_2) \leftarrow & H_0(A) & \leftarrow H_1(X) \\
0 & 1 & n_1 + n_2 & n & k \\
0 & 1 & (n_1+n_2) - 1 & n - (n_1+n_2) + 1 &
\end{array}
$$

$$
\begin{array}{|c|c|c|c}
\leftarrow H_1(X_1) + H_1(X_2) \leftarrow & H_1(A) \leftarrow & H_2(X) \leftarrow & 0 \\
2n - (n_1 + n_2) & n & 1 & 0 \\
n - 1 & 1 & 0 & 0
\end{array}
$$

The missing entry in the middle of the third row can be computed by applying the rule from either side, obtaining $k - [n - (n_1+n_2)+1]$ from the left and $n-(n_1+n_2)+1$ from the right. However, since ϕ_1 is the zero map, this entry must be zero. Combining these gives $k=0$, so that $H_1(X)=0$ and hence X is a 2-sphere.

It follows from Lemma 2 that if M is as in Lemma 1 and M is not a 2-sphere, then there is an annulus A lying in $M \cap \alpha_0$ such that neither of the boundary curves of A is homologous to zero on M. The next lemma is designed to show that if such an annulus is removed from M and replaced by a pair of suitably chosen disks, the result will be a 2-manifold of genus less than M, or the union of two such manifolds.

LEMMA 3. *If M is a compact, connected, orientable 2-manifold of genus p, A is an annulus on M bounded by simple closed curves J_1 and J_2 which are not homologous to zero on M and D_1 and D_2 are disjoint disks such that $D_1 \cap M = J_1$ and $D_2 \cap M = J_2$, then $(M-A) \cup D_1 \cup D_2$ is either a connected 2-manifold of genus $p-1$ or the union of two disjoint connected 2-manifolds each having genus less than p.*

PROOF. Let $X_1 = D_1 \cup D_2$, $X_2 = M - A$ and $X = (M-A) \cup D_1 \cup D_2 = X_1 \cup X_2$. Then $(M; A, X_2)$ is a proper triad so the Meyer-Vietoris formula $\chi(M) = \chi(A) + \chi(X_2) - \chi(A \cap X_2)$, relating the Euler characteristics of M, A and X_2 holds [2, p. 53]. Let p denote the genus of M. Then the above formula gives $\chi(X_2) = 2 - 2p$. Also, $(X; X_1, X_2)$ is a proper triad, so $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$, from which it follows that $\chi(X) = 4 - 2p$. If X is connected and q is the genus of X, then $\chi(X) = 2 - 2q$; equating these two values of $\chi(X)$, we obtain

$q=p-1$. If X is not connected, then it is the union of two mutually exclusive closed surfaces Y_1 and Y_2. Let q_i $(i=1, 2)$ denote the genus of Y_i. Then $\chi(Y_i)=2-2q_i$, and since $4-2p=\chi(X)=\chi(Y_1)+\chi(Y_2)=4-2(q_1+q_2)$, $q_1+q_2=p$. If $q_i=0$, then Y_i is a 2-sphere and $\mathrm{Cl}(Y_i-D_i)$ is a disk on M bounded by J_i; but this is impossible since J_i is not homologous to zero on M. Hence q_1 and q_2 are both positive, so each is less than p.

Thus if M is as in Lemma 1 and is not a 2-sphere, it can be modified by the deletion of an annulus and the addition of two disks so as to obtain either one or two disjoint 2-manifolds, each of which is "more like" a 2-sphere than M, in the sense of having a smaller genus. A finite sequence of such modifications will change M into the sum of disjoint 2-spheres; the next task is to show that these modifications can be carried out in such an order as not to interfere with each other and to end up with a set of 2-spheres satisfying the desired conditions.

Let $I_1, I_2, \cdots, I_n$ be the components of $I\cap U_0$. Then if $i\neq j$, $\bar{I}_i\cap\bar{I}_j=\emptyset$ and for each i, there is a connected, polyhedral 2-manifold M_i satisfying, with respect to I_i, all the conditions of Lemma 1.

Let η be a positive number such that no point of F is within η of any point of $C_1\cup C_2$, and replace C_2 by the cube C_2' obtained as the union of $\alpha_{-\eta}\cap(C_2\cup\mathrm{Int}\ C_2)$ and $C_2\cap(E^3-U_{-\eta})$. (Recall that for each real number t, α_t is the plane $z=t$ and U_t is the set of all points of E^3 lying above α_t.)

Any component of $M_i\cap\alpha_0$ is a disk with a finite number of holes; if D is such a component, then since every simple closed curve on D is homologous to a linear combination of boundary curves of D, it follows that if there is any simple closed curve on D which is not homologous to zero on M_j (where $D\subset M_j$), then the boundary of one of the bounded components of α_0-D is not homologous to zero on M_j.

Suppose now that not every M_i is a 2-sphere. Then by Lemma 2, there is an annulus A lying in $M_j\cap\alpha_0$ for some j, such that the boundary curves of A are not homologous to zero in M_j; by the preceding remark, A can be chosen so that its inner boundary (i.e., the boundary of the bounded component of α_0-A) is an inner boundary curve of some component of $M_j\cap\alpha_0$. Hence there is an annulus A_1 lying in $(\bigcup M_i)\cap\alpha_0$ such that if Q_1 is the bounded component of α_0-A_1, then every simple closed curve (if any) in $(\bigcup M_i)\cap Q_1$ is homologous to zero on $\bigcup M_i$.

Let M_1 be the one of $M_1, M_2, \cdots, M_n$ which contains A_1. Let J_1 and J_1' be the boundary curves of A_1, with J_1 enclosing J_1' in α_0. Let t_1 and t_1' be numbers such that $-\eta<t_1<t_1'<0$. There exist two mutually exclusive disks D_1 and D_1' bounded by J_1 and J_1', respectively, such that D_1 is the union of a disk in α_{t_1} and part of the vertical cylinder over J_1 and D_1' is the union of a disk in $\alpha_{t_1'}$ and part of the vertical cylinder over J_1'. Let $M_j' = (M_j - A_1) \cup (D_1 \cup D_1')$. Then by Lemma 3, M_j' is either a connected 2-manifold of genus less than that of M_j or the union of two disjoint such 2-manifolds. Furthermore, $M_j' \subset \eta$-neighborhood $(C_1 \cup C_2')$, $M_j' \cap F = \varnothing$, and the interior, I_j', of M_j' contains I_j.

For each $i \neq j$, let $M_i' = M_i$, and suppose that $\bigcup_{i=1}^{n} M_i'$ is not the union of disjoint 2-spheres. Then (again using Lemma 2), there is an annulus A lying in some M_i' such that the boundary curves of A are not homologous to zero in $\bigcup M_i'$. Since Q_1 was chosen so that every simple closed curve in $(\bigcup M_i) \cap Q_1$ is homologous to zero in $\bigcup M_i$, A is not contained in Q_1. Hence A is a subset of a component D of $(\bigcup M_i) \cap \alpha_0$ which misses Q_1; as before, we can choose an annulus A_2 lying in D such that A_2 is not homologous to zero on $\bigcup M_i'$ and such that if Q_2 is the bounded component of $\alpha_0 - A_2$, then every simple closed curve in $(\bigcup M_i') \cap Q_2$ is homologous to zero in $\bigcup M_i'$. It follows that either Q_1 and Q_2 are mutually exclusive or $Q_1 \subset Q_2$. Let J_2 and J_2' be the boundary curves of A_2, with J_2 enclosing J_2' in α_0. Let t_2 and t_2' be numbers such that $-\eta < t_2 < t_2' < t_1$. There exist two mutually exclusive disks D_2 and D_2' bounded by J_2 and J_2', respectively, such that $D_2(D_2')$ is the union of a disk in $\alpha_{t_2}(\alpha_{t_2'})$ and part of the vertical cylinder over $J_2(J_2')$. It follows that $D_2 \cup D_2'$ does not intersect $D_1 \cup D_1'$ and hence if $A_2 \subset M_k'$ and $M_k'' = (M_k' - A_2) \cup (D_2 \cup D_2')$, then M_k'' is the union of a finite number of disjoint 2-manifolds each of genus less than that of M_k'. Also, M_k'' is contained in the η-neighborhood of $C_1 \cup C_2'$, $M_k'' \cap F = \varnothing$ and the interior, I_k'', of M_k'' contains I_k'.

A finite number of repetitions of this process will yield a collection of sets $M_1^{(h)}, M_2^{(h)}, \cdots, M_n^{(h)}$, each of which is the union of a finite number of 2-spheres, such that the sum of the interiors of these 2-spheres contains $\bigcup I_i$ and such that for each i, $M_i^{(h)}$ is contained in the η-neighborhood of $C_1 \cup C_2'$. If η is chosen to be less than the δ obtained at the beginning of the proof of Theorem 1, then the inverses under f of the 2-spheres contained in $C_2' \cup (\bigcup M_i^{(h)})$ will be a set of 2-spheres satisfying all the desired conditions.

THEOREM 2. *If G is a finite collection of 2-spheres in E^3, E is a closed subset of the sum of their interiors which does not intersect any of them, and ϵ is a positive number, then there exists a finite collection H of mutually exclusive 2-spheres with mutually exclusive interiors such that E is a subset of the sum of the interiors of the members of H and each member of H lies in the ϵ-neighborhood of the sum of the members of G.*

PROOF. Suppose n is a positive integer such that the desired result holds for any collection of n 2-spheres, and let G, E and ϵ satisfy the hypothesis of the theorem, with G consisting of $n+1$ 2-spheres $Q_1, \cdots, Q_{n+1}$. Let $G' = \{Q_1, \cdots, Q_n\}$ and let E' denote the part of E lying in the sum of the interiors of $Q_1, \cdots, Q_n$. It follows from the induction hypothesis that there exists a finite collection H' of 2-spheres satisfying the conclusion of Theorem 2 with respect to G', E' and δ, where δ is a positive number less than $\epsilon/2$ and also less than the distance from E to $E-E'$. Then if P_1 denotes the union of the members of H and $P_2 = Q_{n+1}$, there exists a finite collection H of 2-spheres satisfying all the conditions of Theorem 1 with respect to P_1, P_2, E and $\epsilon/2$. Since every member of H is contained in the $\epsilon/2$-neighborhood of $P_1 \cup P_2$ and $P_1 \cup P_2$ is contained in the $\epsilon/2$-neighborhood of the sum of the members of G, H satisfies all the desired conditions.

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