ON THE SIMULTANEOUS SOLUTION OF A CERTAIN SYSTEM OF LINEAR INEQUALITIES

GEORGE J. MINTY

The author recently proved the following theorem:

**Theorem 1.** Let $X$ be a Hilbert space, with real or complex scalars and inner product $(x, y)$. Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ be given such that

$$\text{Re}(x_i - x_j, y_i - y_j) \geq 0 \quad (i, j = 1, \ldots, m),$$

and let $x$ be any point of $X$. Then there exists a point $y$ such that

$$\text{Re}(x_i - x, y_i - y) \geq 0 \quad (i = 1, \ldots, m).$$

The proof was patterned after Schoenberg's [3] proof of Kirszbraun’s theorem. B. Grünbaum [2] has generalized my and Schoenberg's proofs to obtain a theorem which incorporates Theorem 1 and Kirszbraun's theorem, and J. G. Wendel has contributed a neater proof of Theorem 1. With Professor Wendel's permission, I reproduce his proof:

**Lemma.** Let $X$ be $E^n$, with the usual (real) scalars and inner product, and let $x_1, \ldots, x_m; y_1, \ldots, y_m$ be given such that

$$\langle x_i - x_j, y_i - y_j \rangle \geq 0 \quad (i, j = 1, \ldots, m).$$

Then there exists a point $y$ such that

$$\langle x_i, y \rangle \leq \langle x_i, y_i \rangle \quad (i = 1, \ldots, m).$$

**Proof of the Lemma.** Let $A$ be the matrix whose $t$th row is $x_t$, and let $b$ be the column-vector whose $i$th element is $\langle x_i, y_i \rangle$. Then $(2')$ is equivalent to $Ay \leq b$. If there is no solution for $y$, then by Stiemke's Theorem [1, Theorem 2.7] there exists a row-vector $\eta \leq 0$ such that

$$\eta A = 0 \quad \text{and} \quad \eta b = 1.$$  

Suppose this to be the case. Then $(3a)$ implies $\sum_i \eta_i x_i = 0$, hence for each $j$,

$$\sum_i \eta_i (x_i, y_j) = 0.$$  

Also $(3b)$ implies

Received by the editors January 11, 1961.
Expanding (1'), multiplying by \( \eta x_i \) and summing on \( i, j \), we have

\[
\sum_i \eta_i \langle x_i, y_i \rangle \sum_j \eta_j + \sum_i \eta_i \sum_j \eta_j \langle x_j, y_j \rangle \geq \sum_i \eta_i \sum_j \eta_j \langle x_j, y_j \rangle + \sum_i \eta_i \sum_j \eta_j \langle x_i, y_i \rangle
\]

i.e., by (4) and (5), \( 2 \sum_j \eta_j \geq 0 \). But \( \eta \leq 0 \) and some \( \eta < 0 \) by (3b). We have a contradiction, and there is at least one solution for \( y \).

Proof of Theorem 1. First, we take up the case where \( X = E^n \), and set \( x'_i = x_i - x \); the conclusion follows by application of the Lemma to \( x'_1, \ldots, x'_m \) and \( y_1, \ldots, y_m \).

We next suppose that \( X \) is any finite-dimensional Hilbert space. If the scalars are complex, it is easily verified that \( \langle x, y \rangle = \text{Re} \langle x, y \rangle \) is a real inner product provided the scalar product \( \alpha \cdot x \) is restricted to real \( \alpha \), and that the resulting Hilbert space with real scalars is of dimension \( 2n \). The conclusion now follows from the isomorphism with \( E^n \) or \( E^{2n} \).

In case \( X \) is infinite-dimensional, we simply apply the above results to the (finite-dimensional) subspace spanned by \( x_1, \ldots, x_m \); \( y_1, \ldots, y_m \) and \( x \).

References


University of Michigan