1. Introduction. If \((p_1, p_2, \ldots, p_n)\) is a permutation of elements 1 to \(n\), then \((\pi_1, \pi_2, \ldots, \pi_n)\) with \(\pi_j = p_j - j \pmod{n}\) is the corresponding difference form. Since \(p_1 + \cdots + p_n = 1 + 2 + \cdots + n\), it follows that \(\pi_1 + \pi_2 + \cdots + \pi_n = 0 \pmod{n}\); hence the difference forms apart from order are partitions of \(kn\), \(k = 0, 1, \ldots, n - 1\) with largest part \(n - 1\) and at most \(n\) parts. Marshall Hall [1] has shown that every such partition corresponds to at least one permutation. Here it is shown that the number of these partitions is given by

\[
P_{0,n} = \frac{1}{n} \sum_{d|n} \phi(n/d) \binom{2d - 1}{d}
\]

with summation over all divisors on \(n\) (including 1 and \(n\)) and \(\phi(n)\) the Euler totient function.

2. A partition enumerator. It is convenient to determine the enumerator for partitions with largest part \(i\) and at most \(n\) parts by use of a theorem of Pólya, as in [4]. Thus they are regarded as unordered arrangements on a line of elements each of which may have any of the values \(0, 1, \ldots, i\) (corresponding to a store enumerator \(1 + x + \cdots + x^n\)) and with order equivalences for all operations of the symmetric group on \(n\) elements. Then, if \(P_n(x, i)\) is the enumerator, by the theorem

\[
P_n(x, i) = S_n(s_1, s_2, \ldots, s_n), \quad s_k = 1 + x^k + \cdots + x^{ik},
\]

with \(S_n(x_1, x_2, \ldots, x_n)\) the cycle index of the symmetric group, which for present purposes may be taken as defined by

\[
\sum_{n=0} S_n(x_1, x_2, \ldots, x_n) y^n = \exp \left( x_1 y + x_2 \frac{y^2}{2} + \cdots + x_n \frac{y^n}{n} + \cdots \right).
\]

Writing

\[
P(x, y) = \sum_{n=0} P_n(x, i) y^n
\]

and using (2) and (3), it is found that

\[
P(x, y) = \frac{1}{(1 - y)(1 - xy) \cdots (1 - x^i y)}
\]

a result which is immediate otherwise. Since

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it follows from (4) that

\[(1 - y)P(x, y) = (1 - x^{i+1}y)P(x, xy)\]

(a result given by P. A. MacMahon [2, p. 5], who has also noticed [2, p. 66] the equivalent result, equation (2). By (2a)

\[P_n(x, n - 1) = P_{n-1}(x, n);\]

by (2), this corresponds to the interesting identity

\[S_n(s_1, s_{n-1}, \ldots, s_{n(n-1)}) = S_{n-1}(s_1, \ldots, s_{n-1});\]

with \(s_k = 1 + x^k + \cdots + x^{ik}\). Notice also that from (2a), on evaluat-

\[P_n(1, n - 1) = \binom{2n - 1}{n}.\]

Finally it may be noticed that the enumerator for compositions is obtained from the theorem as

\[C_n(x, i) = (1 + x + \cdots + x^i)^n\]

since the group of equivalences consists solely of the identity (cycle index \(x^n_i\)).

3. Multisection of enumerators. The enumerator \(P_n(x, n - 1)\) gives

as coefficient of \(x^m, m = 0, 1, \ldots, n(n-1)\), the number of partitions

of \(m\) into at most \(n\) parts and with largest part \(n-1\). The partitions

corresponding to permutations in difference form are for only those

values of \(m\) which are zero or multiples of \(n\). To pick out such terms

requires what DeMorgan [3] calls multisection of the series of terms

in the enumerator, which is accomplished by simple properties of the

roots of unity. Briefly if

\[a(x) = a_0 + a_1x + \cdots\]

and \(\alpha\) is a primitive \(n\)th root of unity, then the \(i\)th \(n\)-sectional series

\[a_{i,n}(x) = a_1 x^i + a_{i+n} x^{i+n} + \cdots\]

is given by

\[a_{i,n}(x) = n^{-1} \sum_{j=1}^{n} \alpha^{-ij}a(\alpha^jx).\]
Applied to the partition enumerator $P_n(x, n-1)$, (8) gives

$$(9) \quad P_{i,n}(x, n-1) = n^{-1} \sum_{i=1}^{n} \alpha^{-ij} P_n(\alpha^i x, n - 1)$$

and in particular

$$(10) \quad P_{i,n} = P_{i,n}(1, n-1) = n^{-1} \sum_{i=1}^{n} \alpha^{-ij} P_n(\alpha^i, n - 1)$$

is the sum of the numbers of partitions of all integers congruent to $i$, modulo $n$.

Equation (9) seems not to have much to offer, but equation (10) does. First it is clear that the powers of $\alpha$ may be classified according to their period; there are $\phi(d)$ powers of period $d$, and, if $\beta_1, \beta_2$ are roots, each of period $d$, $P_n(\beta_1, n-1) = P_n(\beta_2, n-1)$. If $\beta^d = 1$ and $de = n$, then

$$s_k(\beta) = 1 + \beta^k + \cdots + \beta^{k(d-1)}$$
$$= (1 + \beta^k + \cdots + \beta^{k(d-1)})(1 + \beta^{kd} + \cdots + \beta^{kd(s-1)})$$

and since $1 + \beta + \cdots + \beta^{d-1} = 0$,

$$(11) \quad s_k(\beta) = 0, \quad d \nmid k,$$
$$= n, \quad d \mid k.$$

Hence, by (2)

$$(12) \quad P_n(\beta, n-1) = S_n(0, \cdots, n, 0, \cdots, n, \cdots), \quad \beta^d = 1,$$

the nonzero entries in $S_n$ occurring at positions $jd, j = 1, 2, \cdots$.

If in (3) $x_k = 0, d \nmid k, x_k = x, d \mid k$, then

$$\sum_{n=0} S_n(x_1, \cdots, x_n) y^n = \exp(x/d) \left( y^d + \frac{y^{2d}}{2} + \cdots \right)$$
$$= (1 - y^d)^{-x/d}$$
$$= \sum_{j=0}^{\infty} \left( \frac{j - 1 + xd^{-1}}{j} \right) y^{jd}.$$

Hence

$$(13) \quad P_n(\beta, n-1) = \left( \frac{2d - 1}{e} \right), \quad \beta^d = 1, \quad de = n,$$

and by (10) with $i = 0,$
\[ P_{0,n} = n^{-1} \sum_{d|n} \phi(d) \left( \frac{2e - 1}{e} \right), \quad de = n, \]

the result stated in the introduction.

The \( P_{i,n} \) may all be expressed in terms of the \( P_{0,n} \). Thus for \( n = p \), a prime,

\[ P_{i,p} = P_{0,p} - 1, \quad i = 1, 2, \ldots, p - 1. \]

For \( n = pq \), \( p \) and \( q \) prime,

\[ P_{i,pq} = P_{0,pq} - P_{0,p} - P_{0,q} + 1, \quad i \leq p, q, \]

\[ P_{jp,pq} = P_{0,pq} - P_{0,p}, \quad j = 1, 2, \ldots, q - 1, \]

\[ P_{jq,pq} = P_{0,pq} - P_{0,q}, \quad j = 1, 2, \ldots, p - 1. \]

For \( n = p^k \),

\[ P_{i,p^k} = P_{0,p^k} - P_{0,p^{k-1}}, \quad j < k. \]

Finally it may be noticed that the corresponding composition sums \( C_{i,n} \) (defined as in (10)) all have the common value

\[ C_{i,n} = n^{n-1}, \]

since \( C_n(\alpha^i, n-1) = 0, j < n \) and \( C_n(1, n-1) = n^n \). Hence they are equinumerous with fully point-labeled rooted trees.

REFERENCES


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