ENUMERATIONS FOR PERMUTATIONS IN DIFFERENCE FORM

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1. **Introduction.** If $(p_1, p_2, \cdots, p_n)$ is a permutation of elements 1 to n, then $(\pi_1, \pi_2, \cdots, \pi_n)$ with $\pi_j \equiv p_j - j \pmod{n}$ is the corresponding difference form. Since $p_1 + \cdots + p_n = 1 + 2 + \cdots + n$, it follows that $\pi_1 + \pi_2 + \cdots + \pi_n \equiv 0 \pmod{n}$; hence the difference forms apart from order are partitions of kn, $k = 0, 1, \cdots, n-1$ with largest part $n-1$ and at most n parts. Marshall Hall [1] has shown that every such partition corresponds to at least one permutation. Here it is shown that the number of these partitions is given by

$$P_{0,n} = \frac{1}{n} \sum_{d|n} \phi(n/d) \binom{2d-1}{d} \tag{1}$$

with summation over all divisors on n (including 1 and n) and $\phi(n)$ the Euler totient function.

2. **A partition enumerator.** It is convenient to determine the enumerator for partitions with largest part i and at most n parts by use of a theorem of Pólya, as in [4]. Thus they are regarded as unordered arrangements on a line of elements each of which may have any of the values $0, 1, \cdots, i$ (corresponding to a *store* enumerator $1 + x + \cdots + x^i$) and with order equivalences for all operations of the symmetric group on n elements. Then, if $P_n(x, i)$ is the enumerator, by the theorem

$$P_n(x, i) = S_n(s_1, s_2, \cdots, s_n), \qquad s_k = 1 + x^k + \cdots + x^{ik}, \tag{2}$$

with $S_n(x_1, x_2, \cdots, x_n)$ the cycle index of the symmetric group, which for present purposes may be taken as defined by

$$\sum_{n=0} S_n(x_1, x_2, \cdots x_n) y^n = \exp\left(x_1 y + x_2 \frac{y^2}{2} + \cdots + x_n \frac{y^n}{n} + \cdots\right). \tag{3}$$

Writing

$$P(x, y) = \sum_{n=0} P_n(x, i) y^n$$

and using (2) and (3), it is found that

$$P(x, y) = 1/(1-y)(1-xy) \cdots (1-x^i y) \tag{4}$$

a result which is immediate otherwise. Since

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$$(1 - y)P(x, y) = (1 - x^{i+1}y)P(x, xy)$$

it follows from (4) that

$$P_n(x, i) = \frac{1 - x^{i+1}}{1 - x} \frac{1 - x^{i+2}}{1 - x^2} \cdots \frac{1 - x^{i+n}}{1 - x^n} \tag{2a}$$

a result given by P. A. MacMahon [2, p. 5], who has also noticed [2, p. 66] the equivalent result, equation (2). By (2a)

$$P_n(x, n - 1) = P_{n-1}(x, n);$$

by (2), this corresponds to the interesting identity

$$S_n(s_{1,n-1}, \cdots, s_{n,n-1}) = S_{n-1}(s_{1,n}, \cdots, s_{n-1,n}) \tag{5}$$

with $s_{k,i} = 1 + x^k + \cdots + x^{ik}$. Notice also that from (2a), on evaluating the indeterminate form,

$$P_n(1, n - 1) = \binom{2n - 1}{n}. \tag{6}$$

Finally it may be noticed that the enumerator for compositions is obtained from the theorem as

$$C_n(x, i) = (1 + x + \cdots + x^i)^n \tag{7}$$

since the group of equivalences consists solely of the identity (cycle index x_1^n).

3. **Multisection of enumerators.** The enumerator $P_n(x, n-1)$ gives as coefficient of x^m, $m = 0, 1, \cdots, n(n-1)$, the number of partitions of m into at most n parts and with largest part $n-1$. The partitions corresponding to permutations in difference form are for only those values of m which are zero or multiples of n. To pick out such terms requires what DeMorgan [3] calls multisection of the series of terms in the enumerator, which is accomplished by simple properties of the roots of unity. Briefly if

$$a(x) = a_0 + a_1x + \cdots$$

and α is a primitive nth root of unity, then the ith n-sectional series

$$a_{i,n}(x) = a_ix^i + a_{i+n}x^{i+n} + \cdots$$

is given by

$$a_{i,n}(x) = n^{-1} \sum_{j=1}^{n} \alpha^{-ij} a(\alpha^j x). \tag{8}$$

Applied to the partition enumerator $P_n(x, n-1)$, (8) gives

$$P_{i,n}(x, n-1) = n^{-1} \sum_{j=1}^{n} \alpha^{-ij} P_n(\alpha^j x, n-1) \tag{9}$$

and in particular

$$P_{i,n} \equiv P_{i,n}(1, n-1) = n^{-1} \sum_{j=1}^{n} \alpha^{-ij} P_n(\alpha^j, n-1) \tag{10}$$

is the sum of the numbers of partitions of all integers congruent to i, modulo n.

Equation (9) seems not to have much to offer, but equation (10) does. First it is clear that the powers of α may be classified according to their period; there are $\phi(d)$ powers of period d, and, if β_1, β_2 are roots, each of period d, $P_n(\beta_1, n-1) = P_n(\beta_2, n-1)$. If $\beta^d = 1$ and $de = n$, then

$$\begin{aligned} s_k(\beta) &= 1 + \beta^k + \cdots + \beta^{k(de-1)} \\ &= (1 + \beta^k + \cdots + \beta^{k(d-1)})(1 + \beta^{kd} + \cdots + \beta^{kd(e-1)}) \end{aligned}$$

and since $1 + \beta + \cdots + \beta^{d-1} = 0$,

$$\begin{aligned} s_k(\beta) &= 0, & d \nmid k, \\ &= n, & d \mid k. \end{aligned} \tag{11}$$

Hence, by (2)

$$P_n(\beta, n-1) = S_n(0, \cdots, n, 0, \cdots n, \cdots), \qquad \beta^d = 1, \tag{12}$$

the nonzero entries in S_n occurring at positions jd, $j = 1, 2, \cdots$.

If in (3) $x_k = 0$, $d \nmid k$, $x_k = x$, $d \mid k$, then

$$\begin{aligned} \sum_{n=0} S_n(x_1, \cdots, x_n) y^n &= \exp(x/d)\left(y^d + \frac{y^{2d}}{2} + \cdots\right) \\ &= (1 - y^d)^{-x/d} \\ &= \sum_{j=0} \binom{j - 1 + xd^{-1}}{j} y^{jd}. \end{aligned} \tag{13}$$

Hence

$$P_n(\beta, n-1) = \binom{2e-1}{e}, \qquad \beta^d = 1, \qquad de = n, \tag{14}$$

and by (10) with $i = 0$,

$$P_{0,n} = n^{-1} \sum_{d|n} \phi(d) \binom{2e-1}{e}, \qquad de = n, \tag{1}$$

the result stated in the introduction.

The $P_{i,n}$ may all be expressed in terms of the $P_{0,n}$. Thus for $n=p$, a prime,

$$P_{i,p} = P_{0,p} - 1, \qquad i = 1, 2, \cdots, p-1. \tag{14}$$

For $n=pq$, p and q prime,

$$\begin{aligned} P_{i,pq} &= P_{0,pq} - P_{0,p} - P_{0,q} + 1, & i \nmid p, q, \\ P_{jp,pq} &= P_{0,pq} - P_{0,p}, & j = 1, 2, \cdots, q-1, \\ P_{jq,pq} &= P_{0,pq} - P_{0,q}, & j = 1, 2, \cdots, p-1. \end{aligned} \tag{15}$$

For $n=p^k$,

$$P_{p^j,p^k} = P_{0,p^k} - P_{0,p^{k-j+1}}, \qquad j < k. \tag{16}$$

Finally it may be noticed that the corresponding composition sums $C_{i,n}$ (defined as in (10)) all have the common value

$$C_{i,n} = n^{n-1}, \tag{17}$$

since $C_n(\alpha^j, n-1)=0$, $j<n$ and $C_n(1, n-1)=n^n$. Hence they are equinumerous with fully point-labeled rooted trees.

References

1. Marshall Hall, Jr., *A combinatorial problem on Abelian groups*, Proc. Amer. Math. Soc. **3** (1952), 548–587.

2. P. A. MacMahon, *Combinatory analysis*, Vol. II, Cambridge Univ. Press, Cambridge, 1916.

3. Augustus DeMorgan, *The differential and integral calculus*, Robert Baldwin, London, 1942.

4. John Riordan, *The combinatorial significance of a theorem of Pólya*, J. Soc. Indust. Appl. Math. **5** (1957), 225–237.

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