scendental and an algebraic function. Thus we reached a contradiction and $P(x)\equiv 0$ which means that $\int_0^1 (P(x))^2 dx = 0$.

Reference

1. B. Nyman, *On the one-dimensional translation group and semi-group in certain function spaces*, Dissertation, Univ. of Uppsala, 1950. Math. Rev. **12** (1951), 108; Zbl. Math. **37** (1951), 354.

Wayne State University

A NOTE ON THE LIPSCHITZ CONDITION IN METRIC SPACES

NORMAN LEVINE

Let f: $M_1 \to M_2$ be a single valued transformation where M_1 and M_2 are metric spaces with metrics d_1 and d_2 respectively.

Definition. f: $M_1 \to M_2$ is termed Lipschitzian if and only if there exists a finite constant K such that $d_2(f(a), f(b)) \leqq K d_1(a, b)$ for all $a, b \in M_1$.

Theorem. *Let $\{M_i, d_i\}$ be an infinite sequence of metric spaces and $f_{i,j,k}$: $M_i \to M_j$ continuous for all positive integers i, j and k. Then for each i there exists a metric d_i^* for M_i which is equivalent to d_i such that $f_{i,j,k}$: $M_i \to M_j$ is Lipschitzian relative to d_i^* and d_j^* for all i, j and k.*

Proof. Without loss of generality we may assume that each d_i is bounded by 1 on M_i. Let $a, b \in M_s$ and define

$$d_s^*(a, b) \equiv d_s(a, b) + \sum_{q=1}^{\infty} 1/2^{q+n_1+\cdots+n_q+k_1+\cdots+k_q}$$
$$\cdot d_{n_1}(f_{n_2,n_1,k_1} \cdots f_{n_q,n_{q-1},k_{q-1}} f_{s,n_q,k_q}(a), \cdots (b)),$$

where the summation runs over all $n_1, \cdots, n_q \in S_q$ and $k_1, \cdots, k_q \in S_q$, S_q being the set of all finite sequences of positive integers with q elements; juxtaposition of the f's of course means composition. Then d_s^* satisfies all of the axioms for a metric on M_s. The proof will be complete when we show that (1) d_s^* is equivalent to d_s on M_s and (2) $f_{i,j,k}$: $M_i \to M_j$ is Lipschitzian relative to d_i^* and d_j^*. To show (1) let $y \in M_s$ and $\{y_n\}$ be an infinite sequence in M_s. Suppose $\lim_n d_s^*(y, y_n) = 0$. Then $\lim_n d_s(y, y_n) = 0$ since $d_s(y, y_n) \leqq d_s^*(y, y_n)$. Conversely suppose $\lim_n d_s(y, y_n) = 0$. We note first that

Received by the editors December 29, 1960, and, in revised form, March 18, 1961.

$$d_s^*(a, b) \leqq 1 + \sum_{q=1}^{\infty} 1/2^{q+n_1+\cdots+n_q+k_1+\cdots+k_q} \quad \text{(summed over } S_q \text{ as above)}$$

for all a, b in M_s. Hence the series representation for $d_s^*(a, b)$ is uniformly convergent, and each term is a continuous function of the arguments a and b. Thus $\lim_n d_s^*(y, y_n) = d_s^*(y, \lim_n y_n) = d_s^*(y, y) = 0$. To show (2) consider $f_{i,j,k}$: $M_i \to M_j$. Take a and b in M_i. Then

$$\begin{aligned} d_j^*(f_{i,j,k}(a), f_{i,j,k}(b)) &= d_j(f_{i,j,k}(a), f_{i,j,k}(b)) \\ &\quad + \sum_{\substack{q=1 \\ n_1,\cdots,n_q \in S_q \\ k_1,\cdots,k_q \in S_q}}^{\infty} 1/2^{q+n_1+\cdots+n_q+k_1+\cdots+k_q} \\ &\quad \cdot d_{n_1}(f_{n_2,n_1,k_1} \cdots f_{j,n_q,k_q} f_{i,j,k}(a), \cdots (b)) \\ &= 2^{1+j+k} \Big\{ 1/2^{1+j+k} d_j(f_{i,j,k}(a), f_{i,j,k}(b)) \\ &\quad + \sum_{\substack{q=1 \\ n_1,\cdots,n_q \in S_q \\ k_1,\cdots,k_q \in S_q}}^{\infty} 1/2^{q+1+n_1+\cdots+n_q+j+k_1+\cdots+k_q+k} \\ &\quad \cdot d_{n_1}(f_{n_2,n_1,k_1} \cdots f_{j,n_q,k_q} f_{i,j,k}(a), \cdots (b)) \Big\} \\ &\leqq 2^{1+j+k} d_i^*(a, b). \end{aligned}$$

Thus $d_j^*(f_{i,j,k}(a), f_{i,j,k}(b)) \leqq K d_i^*(a, b)$ where $K = 2^{1+j+k}$ and the proof is complete.

That the theorem has no generalization to the uncountable case is evident from the following example:

Let M_1 be the space of rationals and M_2 be the interval $[0, 1]$, both with the usual topology. Let d_1 and d_2 be metrics for M_1 and M_2 respectively (not necessarily the natural metrics). Now $\{M_1, d_1\}$ is not complete since any complete, dense-in-itself space is uncountable. Suppose further that $\{f_\alpha\}$ is the class of all continuous mappings from M_1 to M_2. We assert that at least one of the f_α is not uniformly continuous (and hence not Lipschitzian) relative to d_1 and d_2. Since $\{M_1, d_1\}$ is not complete we may take $\{x_k\}$, a Cauchy sequence of distinct points which is not convergent. Let $F \equiv \bigcup \{x_{2n}\}$ and $G \equiv \bigcup \{x_{2n-1}\}$. F and G are nonempty, closed, and disjoint. By Urysohn's Lemma there exists a continuous map g: $M_1 \to M_2$ such that $g(F) = 0$ and $g(G) = 1$. Then $g \in \{f_\alpha\}$. But g: $M_1 \to M_2$ is not uniformly continuous since $d_1(x_{2n}, x_{2n-1})$ converges to 0, but $d_2(g(x_{2n}), g(x_{2n-1}))$ does not converge to 0.

OHIO STATE UNIVERSITY