scendental and an algebraic function. Thus we reached a contradiction and $P(x) = 0$ which means that $\int_0^a (P(x))^2 \, dx = 0$.

REFERENCE


WAYNE STATE UNIVERSITY

A NOTE ON THE LIPSCHITZ CONDITION IN METRIC SPACES

NORMAN LEVINE

Let $f: M_1 \to M_2$ be a single valued transformation where $M_1$ and $M_2$ are metric spaces with metrics $d_1$ and $d_2$ respectively.

DEFINITION. $f: M_1 \to M_2$ is termed Lipschitzian if and only if there exists a finite constant $K$ such that $d_2(f(a), f(b)) \leq K d_1(a, b)$ for all $a, b \in M_1$.

THEOREM. Let $\{ M_i, d_i \}$ be an infinite sequence of metric spaces and $f_{i,j,k}: M_i \to M_j$ continuous for all positive integers $i$, $j$ and $k$. Then for each $i$ there exists a metric $d_i^*$ for $M_i$ which is equivalent to $d_i$ such that $f_{i,j,k}: M_i \to M_j$ is Lipschitzian relative to $d_i^*$ and $d_j^*$ for all $i$, $j$ and $k$.

PROOF. Without loss of generality we may assume that each $d_i$ is bounded by 1 on $M_i$. Let $a, b \in M_i$ and define

$$d_i^*(a, b) = d_i(a, b) + \sum_{q=1}^{\infty} \frac{1}{2^{q+n_1+k_1} \cdots + n_q+k_q+\cdots+k_q}$$

$$\cdot d_n(f_{n_1,n_1,k_1} \cdots f_{n_q,n_q-k_q-1,k_q-1} f_{i,q,q}(a), \cdots (b)),$$

where the summation runs over all $n_1, \cdots, n_q \in S_q$ and $k_1, \cdots, k_q \in S_q$, $S_q$ being the set of all finite sequences of positive integers with $q$ elements; juxtaposition of the $f$'s of course means composition. Then $d_i^*$ satisfies all of the axioms for a metric on $M_i$. The proof will be complete when we show that (1) $d_i^*$ is equivalent to $d_i$ on $M_i$ and (2) $f_{i,j,k}: M_i \to M_j$ is Lipschitzian relative to $d_i^*$ and $d_j^*$. To show (1) let $y \in M_i$ and $\{ y_n \}$ be an infinite sequence in $M_i$. Suppose $\lim_n d_i^*(y, y_n) = 0$. Then $\lim_n d_i(y, y_n) = 0$ since $d_i(y, y_n) \leq d_i^*(y, y_n)$. Conversely suppose $\lim_n d_i(y, y_n) = 0$. We note first that

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for all \( a, b \) in \( M \). Hence the series representation for \( d^*_i(a, b) \) is uniformly convergent, and each term is a continuous function of the arguments \( a \) and \( b \). Thus \( \lim_n d^*_i(y_n) = d^*_i(y, \lim_n y_n) = d^*_i(y, y) = 0 \).

To show (2) consider \( f_{i,j,k}: M_i \to M_j \). Take \( a \) and \( b \) in \( M_i \). Then

\[
d^*_i(f_{i,j,k}(a), f_{i,j,k}(b)) = d^*_i(f_{i,j,k}(a), f_{i,j,k}(b)) + \sum_{q=1}^{\infty} 1/2^{q+n_1+\cdots+n_q+k_1+\cdots+k_q} \cdot d_n(f_{n_1,n_1,k_1} \cdots f_{i,n_q,k_q} f_{i,j,k}(a), \cdots (b))
\]

\[
= 2^{1+i+k} \left\{ 1/2^{1+i+k} d^*_i(f_{i,j,k}(a), f_{i,j,k}(b)) + \sum_{q=1}^{\infty} 1/2^{q+i+n_1+\cdots+n_q+i+k_1+\cdots+k_q+k} \cdot d_n(f_{n_1,n_1,k_1} \cdots f_{i,n_q,k_q} f_{i,j,k}(a), \cdots (b)) \right\}
\]

\[
\leq 2^{1+i+k} d^*_i(a, b).
\]

Thus \( d^*_i(f_{i,j,k}(a), f_{i,j,k}(b)) \leq Kd^*_i(a, b) \) where \( K = 2^{1+i+k} \) and the proof is complete.

That the theorem has no generalization to the uncountable case is evident from the following example:

Let \( M_1 \) be the space of rationals and \( M_2 \) be the interval \([0, 1]\), both with the usual topology. Let \( d_1 \) and \( d_2 \) be metrics for \( M_1 \) and \( M_2 \) respectively (not necessarily the natural metrics). Now \( \{M_1, d_1\} \) is not complete since any complete, dense-in-itself space is uncountable. Suppose further that \( \{f_a\} \) is the class of all continuous mappings from \( M_1 \) to \( M_2 \). We assert that at least one of the \( f_a \) is not uniformly continuous (and hence not Lipschitzian) relative to \( d_1 \) and \( d_2 \). Since \( \{M_1, d_1\} \) is not complete we may take \( \{x_k\} \), a Cauchy sequence of distinct points which is not convergent. Let \( F = \cup \{x_{2n}\} \) and \( G = \cup \{x_{2n-1}\} \). \( F \) and \( G \) are nonempty, closed, and disjoint. By Urysohn’s Lemma there exists a continuous map \( g: M_1 \to M_2 \) such that \( g(F) = 0 \) and \( g(G) = 1 \). Then \( g \in \{f_a\} \). But \( g: M_1 \to M_2 \) is not uniformly continuous since \( d_1(x_{2n}, x_{2n-1}) \) converges to 0, but \( d_2(g(x_{2n}), g(x_{2n-1})) \) does not converge to 0.

Ohio State University