ON A PROBLEM IN THE THEORY OF PARTITIONS

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In this note we give an affirmative answer to a problem posed by Sherman K. Stein (Bull. Amer. Math. Soc. 66 (1960), 510). That is, we show that for every number \( m \), \( 1 \leq m \leq p(n) \) where \( p(n) \) is the number of partitions of the positive integer \( n \), there exists a set \( A(m, n) \) of positive integers such that \( n \) has \( m \) partitions into elements of \( A(m, n) \). Our result generalizes to the case of partitions of vectors with nonnegative integral components.

Definition 1. Let \( S = \{ s_1, s_2, \ldots, s_k \} \) be a set of positive integers; then \( p(S, n) \) denotes the number of partitions of \( n \) into elements of \( S \). In particular we define \( p(S, 0) = 1 \) and \( p(S, n) = 0 \) for \( n < 0 \).

Lemma 1.

\[
p(S, n) = \sum_{i=1}^{k} p(S - s_i, n - s_i)
\]

where \( S_i = \{ s_1, s_2, \ldots, s_{i-1} \} \), \( S_1 = \emptyset \). (Note: this lemma is independent of the ordering of \( S \).)

Proof. The \( i \)th term in the sum counts the number of partitions of \( n \) into elements of \( S \) where the first term occurring is \( s_i \).

Corollary 1. If \( 1 \in S \) then \( p(S, n) \) is a nondecreasing function of \( n \).

Corollary 2.

\[
p(S, n) \leq \sum_{l=0}^{n-1} p(S, l).
\]

Proof. By Lemma 1

\[
p(S, n) = \sum_{i=1}^{k} p(S - s_i, n - s_i)
\]

\[
\leq \sum_{i=1}^{k} p(S, n - s_i)
\]

\[
\leq \sum_{l=0}^{n-1} p(S, l).
\]

Received by the editors January 9, 1961.

1 This work has been supported in part by a grant from the National Science Foundation.
Corollary 3. If $1 \in S$ then $p(S, n+1) \leq p(S, n) + p(S, n-1)$.

Proof. We use simultaneous induction on $n$ and $k$, the number of elements of $S$. If $n = 0$ then

$$p(S, 1) = 1 = p(S, 0) = p(S, 0) + p(S, -1)$$

for all $S$. If $k = 1$ then $S = \{1\}$ and

$$p(\{1\}, n + 1) = 1 = p(\{1\}, n) \leq p(\{1\}, n) + p(\{1\}, n - 1)$$

for all nonnegative $n$.

Now assume we had the minimal $n$ and a minimal $S$ for which the result is false then there is an $s \in S$, $s \neq 1$ and by Lemma 1

$$p(S, n + 1) = p(S, n + 1 - s) + p(S - \{s\}, n + 1)$$

$$\leq p(S, n - s) + p(S, n - 1 - s) + p(S - \{s\}, n)$$

$$+ p(S - \{s\}, n - 1)$$

$$\leq p(S, n) + p(S, n - 1).$$

Lemma 2. If $1 = a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots$ and $a_{k+1} \leq 2a_k$ where the $a_k$ are integers, then for every integer $n$ with $1 \leq n \leq a_1 + a_2 + \cdots + a_k$ there exist indices $1 \leq i_1 < i_2 < \cdots < i_j \leq k$ so that $n = a_{i_1} + \cdots + a_{i_j}$.

Proof. By induction on $k$. If $k = 1$ then $n = 1 = a_1$. Assume the lemma proved for $k - 1$. If $n \leq a_1 + \cdots + a_{k-1}$ then the lemma is true by the induction hypothesis. If $n > a_1 + \cdots + a_{k-1}$ then

$$n > a_k \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) = a_k \left(1 - \frac{1}{2^{k-1}}\right) \geq a_k - 1$$

so that $n \geq a_k$. Thus $0 \leq n - a_k \leq a_1 + \cdots + a_{k-1}$ and the lemma holds, by the induction hypothesis.

Lemma 3. If $1 \in S$ then the sequence $p(S, 1), p(S, 2), \cdots, p(S, k), \cdots$ satisfies the hypothesis of Lemma 2.

Proof. We have $p(S, 1) = 1$ and $p(S, k+1) \geq p(S, k)$ by Corollary 1; finally $p(S, k+1) \leq 2p(S, k)$ by Corollary 3 of Lemma 1.

Lemma 4. If $S \subset \{1, 2, \cdots, \lfloor n/2 \rfloor\}$ and $T \subset \{\lfloor n/2 \rfloor + 1, \cdots, n\}$ then $p(S \cup T, n) = p(S, n) + \sum_{t \in T} p(S, n - t)$.

Proof. Since $t + t' > n$ for any $t, t' \in T$ a partition of $n$ into elements of $S \cup T$ is either a partition into elements of $S$ alone or $t$ plus a partition of $n - t$ into elements of $S$.

Corollary. If $1 \in S$ and $S \subset \{1, 2, \cdots, \lfloor n/2 \rfloor\}$ then for any integer $m$ with
\begin{align*}
\rho(S, n) & \leq m \\
& \leq \rho(S, n) + \sum_{l=0}^{(n-1)/2} \rho(S, l)
\end{align*}

there exists a set \( T \subseteq \{ \lfloor n/2 \rfloor + 1, \ldots, n \} \) so that \( \rho(S \cup T, n) = m \).

**Lemma 5.**

\[ \rho \left( S \cup \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\}, n \right) \leq 1 + \rho(S, n) + \sum_{l=0}^{(n-1)/2} \rho(S, l). \]

**Proof.** If \( \lfloor n/2 \rfloor \in S \) the result is obvious. If \( \lfloor n/2 \rfloor \notin S \) then

\[ \rho \left( S \cup \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\}, n \right) = \rho(S, n) + \rho \left( S, n - \left\lfloor \frac{n}{2} \right\rfloor \right) + \rho \left( S, n - 2 \left\lfloor \frac{n}{2} \right\rfloor \right) \]

so that the lemma reduces to

\[ \rho(S, m + 1) \leq \sum_{l=0}^{m} \rho(S, l), \quad m = \left\lfloor \frac{n - 1}{2} \right\rfloor; \]

which is Corollary 2 of Lemma 1.

**Lemma 6.** If \( 1 \in S \) and \( S \subseteq \{ 1, 2, \ldots, m-1 \} \) where \( m < \lfloor n/2 \rfloor \) then

\[ \rho(S \cup \{ m \}, n) \leq 1 + \rho \left( S \cup \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\}, n \right) + \sum_{l=0}^{(n-1)/2} \rho \left( S \cup \left\{ m + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}, l \right) \]

\[ = 1 + \rho(S \cup \{ m + 1, \ldots, n \}, n). \]

**Proof.** The hypothesis implies \( m \geq 2 \). By repeated application of Lemma 1 we get

\[ \rho(S \cup \{ m \}, n) = \sum_{l=0}^{\lfloor n/m \rfloor} \rho(S, n - lm) \]

and by Corollary 3 of Lemma 1 together with Lemma 1
\[
\sum_{l=0}^{\left\lceil n/m \right\rceil} p(S, n - lm) 
\leq p \left( S, n - \left\lceil \frac{n}{m} \right\rceil m \right) + p(S, n) + \sum_{l=1}^{\left\lceil n/m \right\rceil-1} p(S, n - lm) 
\leq p \left( S, n - \left\lfloor \frac{n}{m} \right\rfloor m \right) + p(S, n) 
+ \sum_{l=1}^{\left\lceil n/m \right\rceil-1} (p(S, n - lm - 1) + p(S, n - lm - 2)) 
\leq 1 + p(S, n) + \sum_{i=0}^{n-m} p(S, l) 
\leq 1 + p(S, n) + \sum_{k=1}^{n-m} p(S \cup \{m + 1, \ldots, m + k\}, n - m - k) 
= 1 + p(S \cup \{m + 1, \ldots, n\}, n).
\]

**Theorem 1.** For every integer \( m, 1 \leq m \leq p(n) \), there exists a set \( S \subseteq \{1, 2, \ldots, n\} \) such that \( p(S, n) = m \).

**Proof.** We construct a sequence \( S_1, S_2, \ldots, S_n \) of subsets of \( \{1, 2, \ldots, \lfloor n/2 \rfloor\} \) as follows:

1. \( S_1 = \{1\}, \quad S_2 = \{1, \lfloor n/2 \rfloor\} \).
2. Let \( r \) be the maximal element \( \leq \lfloor n/2 \rfloor \) which is not contained in \( S_i \) then \( S_{i+1} = S_i \cup \{r\} - \{r+1, \ldots, \lfloor n/2 \rfloor\} \).

It is clear that this sequence terminates only with \( S_n = \{1, 2, \ldots, \lfloor n/2 \rfloor\} \). According to the corollary to Lemma 4 the numbers \( p(S_i \cup T, n) \), where \( T \subseteq \{\lfloor n/2 \rfloor + 1, \ldots, n\} \), fill the interval \( I_i \) of numbers \( p(S_i, n) \leq m \leq p(S_i \cup \{\lfloor n/2 \rfloor + 1, \ldots, n\}, n) \). By Lemmas 5 and 6 the union \( J_i = I_1 \cup I_2 \cup \cdots \cup I_i \) is itself an interval so that \( J_n \) contains all \( m \) with \( 1 \leq m \leq p(n) \).

One might ask whether our theorem has an analogue if \( p(n) \) is replaced by \( p(S, n) \), that is whether for any \( m \) with \( 1 \leq m \leq p(S, n) \) there is a subset \( S' \) of \( S \) so that \( p(S', n) = m \). The answer is obviously no, since \( p(\{1, 2\}, n) = \lfloor n/2 \rfloor + 1 \) while \( p(\{1\}, n) = 1 \) and \( p(\{2\}, n) \leq 1 \). It is, however, possible to generalize our result to the partition of vectors.

**Definition.** Let \( n = (n_1, n_2, \ldots, n_k) \) where the \( n_i \) are integers; then \( p_k(n) \) is the number of partitions of \( n \) into \( k \)-vectors whose components are nonnegative integers not all of which are zero.

**Theorem 2.** Let \( m \) be an integer, \( 1 \leq m \leq p_k(n) \) then there exists a set
$S$ of nonnegative nonzero $k$-vectors such that the number $p_k(S, n)$ of partitions of $n$ into elements of $S$ is $m$.

The proof involves only simple modifications of the steps which led to Theorem 1. We outline them here. Lemma 1 remains valid. In Corollary 1 the condition $1 \in S$ is to be replaced by the condition that $S$ contains all the unit-vectors. By "nondecreasing" we now refer to the partial ordering given by $a < b$ if $b - a$ has nonnegative components. In Corollary 2 the summation is understood to be over all vectors which precede $n$ in the above sense.

In Corollary 3 the vectors $n+1$ and $n-1$ must be understood as the result of adding and subtracting from $n$ the same (arbitrary) unit vector. We need a slight modification of this result:

If $S$ contains all unit vectors and $n+1 = n'+1'$ where $1$ and $1'$ are two different unit vectors then

$$p_k(S, n + 1) \leq p_k(S, n) + p_k(S, n').$$

Instead of the subsequent lemmas we get

**Lemma 3'.** If $S$ contains the unit vectors then the set $\{p_k(S, t)\}$ where $t$ ranges over all nonnegative vectors with $l < n$ can be arranged as a non-decreasing sequence which satisfies the hypothesis of Lemma 2.

**Lemma 4'.** If $S \cap T = \emptyset$ and $2t > n$ for every $t \in T$, then

$$p_k(S \cup T, n) = p_k(S, n) + \sum_{t \in T} p_k(S, n - t).$$

**Corollary.** Let $S$ contain the unit vectors and $S \cap T = \emptyset$ where $T$ is the set of all vectors $t$ which satisfy $2t > n$. Then for every integer $m$ with $p_k(S, n) \leq m \leq p_k(S \cup T, n)$ there exists a $T' \subseteq T$ so that $p_k(S \cup T', n) = m$.

**Lemma 5'.** Let $n^{(i)} = (n_1, n_2, \ldots, [n_i/2], \ldots, n_k)$ and let $T$ be as in the preceding corollary. Then

$$p_k(S \cup \{n^{(i)}\}, n) \leq 1 + p_k(S \cup T, n).$$

**Lemma 6'.** Let $T$ be as above and let $S$ contain the unit vectors with $S \cap T = \emptyset$. Let $m \in S \cup T$ be a vector all whose successors are contained in $S \cup T$. Let $S_m$ denote the set of successors of $m$. Then

$$p_k(S \cup \{m\} - S_m, n) \leq 1 + p_k(S \cup T, n).$$

None of the proofs involves any new ideas and we therefore omit them. Theorem 2 now follows by exactly the same argument that led to Theorem 1.

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