ON A PROBLEM IN THE THEORY OF PARTITIONS[1]

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In this note we give an affirmative answer to a problem posed by Sherman K. Stein (Bull. Amer. Math. Soc. **66** (1960), 510). That is, we show that for every number m, $1 \leqq m \leqq p(n)$ where $p(n)$ is the number of partitions of the positive integer n, there exists a set $A(m, n)$ of positive integers such that n has m partitions into elements of $A(m, n)$. Our result generalizes to the case of partitions of vectors with nonnegative integral components.

DEFINITION 1. Let $S = \{s_1, s_2, \cdots, s_k\}$ be a set of positive integers; then $p(S, n)$ denotes the number of partitions of n into elements of S. In particular we define $p(S, 0) = 1$ and $p(S, n) = 0$ for $n < 0$.

LEMMA 1.

$$p(S, n) = \sum_{i=1}^{k} p(S - S_i, n - s_i)$$

where $S_i = \{s_1, s_2, \cdots, s_{i-1}\}$, $S_1 = \varnothing$. (*Note: this lemma is independent of the ordering of* S.)

PROOF. The ith term in the sum counts the number of partitions of n into elements of S where the first term occurring is s_i.

COROLLARY 1. *If* $1 \in S$ *then* $p(S, n)$ *is a nondecreasing function of* n.

COROLLARY 2.

$$p(S, n) \leqq \sum_{l=0}^{n-1} p(S, l).$$

PROOF. By Lemma 1

$$\begin{aligned} p(S, n) &= \sum_{i=1}^{k} p(S - S_i, n - s_i) \\ &\leqq \sum_{i=1}^{k} p(S, n - s_i) \\ &\leqq \sum_{l=0}^{n-1} p(S, l). \end{aligned}$$

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COROLLARY 3. *If* $1 \in S$ *then* $p(S, n+1) \leqq p(S, n) + p(S, n-1)$.

PROOF. We use simultaneous induction on n and k, the number of elements of S. If $n=0$ then

$$p(S, 1) \leqq 1 = p(S, 0) = p(S, 0) + p(S, -1)$$

for all S. If $k=1$ then $S=\{1\}$ and

$$p(\{1\}, n+1) = 1 = p(\{1\}, n) \leqq p(\{1\}, n) + p(\{1\}, n-1)$$

for all nonnegative n.

Now assume we had the minimal n and a minimal S for which the result is false then there is an $s \in S$, $s \neq 1$ and by Lemma 1

$$\begin{aligned} p(S, n+1) &= p(S, n+1-s) + p(S - \{s\}, n+1) \\ &\leqq p(S, n-s) + p(S, n-1-s) + p(S - \{s\}, n) \\ &\quad + p(S - \{s\}, n-1) \\ &\leqq p(S, n) + p(S, n-1). \end{aligned}$$

LEMMA 2. *If* $1 = a_1 \leqq a_2 \leqq \cdots \leqq a_k \leqq \cdots$ *and* $a_{k+1} \leqq 2a_k$ *where the* a_k *are integers, then for every integer* n *with* $1 \leqq n \leqq a_1 + a_2 + \cdots + a_k$ *there exist indices* $1 \leqq i_1 < i_2 < \cdots < i_j \leqq k$ *so that* $n = a_{i_1} + \cdots + a_{i_j}$.

PROOF. By induction on k. If $k=1$ then $n=1=a_1$. Assume the lemma proved for $k-1$. If $n \leqq a_1 + \cdots + a_{k-1}$ then the lemma is true by the induction hypothesis. If $n > a_1 + \cdots + a_{k-1}$ then

$$n > a_k\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) = a_k\left(1 - \frac{1}{2^{k-1}}\right) \geqq a_k - 1$$

so that $n \geqq a_k$. Thus $0 \leqq n - a_k \leqq a_1 + \cdots + a_{k-1}$ and the lemma holds, by the induction hypothesis.

LEMMA 3. *If* $1 \in S$ *then the sequence* $p(S, 1), p(S, 2), \cdots, p(S, k), \cdots$ *satisfies the hypothesis of Lemma* 2.

PROOF. We have $p(S, 1) = 1$ and $p(S, k+1) \geqq p(S, k)$ by Corollary 1; finally $p(S, k+1) \leqq 2p(S, k)$ by Corollary 3 of Lemma 1.

LEMMA 4. *If* $S \subset \{1, 2, \cdots, [n/2]\}$ *and* $T \subset \{[n/2]+1, \cdots, n\}$ *then* $p(S \cup T, n) = p(S, n) + \sum_{t \in T} p(S, n-t)$.

PROOF. Since $t + t' > n$ for any $t, t' \in T$ a partition of n into elements of $S \cup T$ is either a partition into elements of S alone or t plus a partition of $n-t$ into elements of S.

COROLLARY. *If* $1 \in S$ *and* $S \subset \{1, 2, \cdots, [n/2]\}$ *then for any integer* m *with*

$$p(S, n) \leqq m$$
$$\leqq p(S, n) + \sum_{l=0}^{[(n-1)/2]} p(S, l)$$

there exists a set $T \subset \{[n/2]+1, \cdots, n\}$ *so that* $p(S \cup T, n) = m$.

LEMMA 5.

$$p\left(S \cup \left\{\left[\frac{n}{2}\right]\right\}, n\right) \leqq 1 + p(S, n) + \sum_{l=0}^{[(n-1)/2]} p(S, l).$$

PROOF. If $[n/2] \in S$ the result is obvious. If $[n/2] \notin S$ then

$$p\left(S \cup \left\{\left[\frac{n}{2}\right]\right\}, n\right)$$
$$= p(S, n) + p\left(S, n - \left[\frac{n}{2}\right]\right) + p\left(S, n - 2\left[\frac{n}{2}\right]\right)$$
$$\leqq 1 + p(S, n) + p\left(S, \left[\frac{n-1}{2}\right] + 1\right)$$

so that the lemma reduces to

$$p(S, m+1) \leqq \sum_{l=0}^{m} p(S, l), \qquad m = \left[\frac{n-1}{2}\right];$$

which is Corollary 2 of Lemma 1.

LEMMA 6. *If* $1 \in S$ *and* $S \subset \{1, 2, \cdots, m-1\}$ *where* $m < [n/2]$ *then*

$$p(S \cup \{m\}, n) \leqq 1 + p\left(S \cup \left\{m+1, \cdots, \left[\frac{n}{2}\right]\right\}, n\right)$$
$$+ \sum_{l=0}^{[(n-1)/2]} p\left(S \cup \left\{m+1, \cdots \left[\frac{n}{2}\right]\right\}, l\right)$$
$$= 1 + p(S \cup \{m+1, \cdots, n\}, n).$$

PROOF. The hypothesis implies $m \geqq 2$. By repeated application of Lemma 1 we get

$$p(S \cup \{m\}, n) = \sum_{l=0}^{[n/m]} p(S, n - lm)$$

and by Corollary 3 of Lemma 1 together with Lemma 1

$$\begin{aligned}
&\sum_{l=0}^{[n/m]} p(S, n - lm) \\
&\quad \leqq p\left(S, n - \left[\frac{n}{m}\right]m\right) + p(S, n) + \sum_{l=1}^{[n/m]-1} p(S, n - lm) \\
&\quad \leqq p\left(S, n - \left[\frac{n}{m}\right]m\right) + p(S, n) \\
&\qquad + \sum_{l=1}^{[n/m]-1} (p(S, n - lm - 1) + p(S, n - lm - 2)) \\
&\quad \leqq 1 + p(S, n) + \sum_{l=0}^{n-m-1} p(S, l) \\
&\quad \leqq 1 + p(S, n) + \sum_{k=1}^{n-m} p(S \cup \{m + 1, \cdots, m + k\}, n - m - k) \\
&\quad = 1 + p(S \cup \{m + 1, \cdots, n\}, n).
\end{aligned}$$

THEOREM 1. *For every integer m, $1 \leqq m \leqq p(n)$, there exists a set $S \subset \{1, 2, \cdots, n\}$ such that $p(S, n) = m$.*

PROOF. We construct a sequence $S_1, S_2, \cdots, S_n$ of subsets of $\{1, 2, \cdots, [n/2]\}$ as follows:

(1) $S_1 = \{1\}$, $S_2 = \{1, [n/2]\}$.

(2) Let r be the maximal element $\leqq [n/2]$ which is not contained in S_i then $S_{i+1} = S_i \cup \{r\} - \{r+1, \cdots, [n/2]\}$.

It is clear that this sequence terminates only with $S_N = \{1, 2, \cdots, [n/2]\}$. According to the corollary to Lemma 4 the numbers $p(S_i \cup T, n)$, where $T \subset \{[n/2]+1, \cdots, n\}$, fill the interval I_i of numbers $p(S_i, n) \leqq m \leqq p(S_i \cup \{[n/2]+1, \cdots, n\}, n)$. By Lemmas 5 and 6 the union $J_i = I_1 \cup I_2 \cup \cdots \cup I_i$ is itself an interval so that J_N contains all m with $1 \leqq m \leqq p(n)$.

One might ask whether our theorem has an analogue if $p(n)$ is replaced by $p(S, n)$, that is whether for any m with $1 \leqq m \leqq p(S, n)$ there is a subset S' of S so that $p(S', n) = m$. The answer is obviously no, since $p(\{1, 2\}, n) = [n/2]+1$ while $p(\{1\}, n) = 1$ and $p(\{2\}, n) \leqq 1$.[2] It is, however, possible to generalize our result to the partition of vectors.

DEFINITION. Let $n = (n_1, n_2, \cdots, n_k)$ where the n_i are integers; then $p_k(n)$ is the number of partitions of n into k-vectors whose components are nonnegative integers not all of which are zero.

THEOREM 2. *Let m be an integer, $1 \leqq m \leqq p_k(n)$ then there exists a set*

[2] The answer is yes if we let $S = \{m, m+1, m+2, \cdots\}$ be the set of all integers $\geqq m$, m a positive integer, without appreciable change in the proof.

S of nonnegative nonzero k-vectors such that the number $p_k(S, n)$ of partitions of n into elements of S is m.

The proof involves only simple modifications of the steps which led to Theorem 1. We outline them here. Lemma 1 remains valid. In Corollary 1 the condition $1 \in S$ is to be replaced by the condition that S contains all the unit-vectors. By "nondecreasing" we now refer to the partial ordering given by $a \prec b$ if $b-a$ has nonnegative components. In Corollary 2 the summation is understood to be over all vectors which precede n in the above sense.

In Corollary 3 the vectors $n+1$ and $n-1$ must be understood as the result of adding and subtracting from n the same (arbitrary) unit vector. We need a slight modification of this result:

If S contains all unit vectors and $n+1=n'+1'$ where 1 and 1′ are two different unit vectors then

$$p_k(S, n+1) \leqq p_k(S, n) + p_k(S, n').$$

Instead of the subsequent lemmas we get

LEMMA 3′. *If S contains the unit vectors then the set $\{p_k(S, l)\}$ where l ranges over all nonnegative vectors with $l \prec n$ can be arranged as a nondecreasing sequence which satisfies the hypothesis of Lemma 2.*

LEMMA 4′. *If $S \cap T = \emptyset$ and $2t \succ n$ for every $t \in T$, then*

$$p_k(S \cup T, n) = p_k(S, n) + \sum_{t \in T} p_k(S, n-t).$$

COROLLARY. *Let S contain the unit vectors and $S \cap T = \emptyset$ where T is the set of all vectors t which satisfy $2t \succ n$. Then for every integer m with $p_k(S, n) \leqq m \leqq p_k(S \cup T, n)$ there exists a $T' \subset T$ so that $p_k(S \cup T', n) = m$.*

LEMMA 5′. *Let $n^{(i)} = (n_1, n_2, \cdots, [n_i/2], \cdots, n_k)$ and let T be as in the preceding corollary. Then*

$$p_k(S \cup \{n^{(i)}\}, n) \leqq 1 + p_k(S \cup T, n).$$

LEMMA 6′. *Let T be as above and let S contain the unit vectors with $S \cap T = \emptyset$. Let $m \notin S \cup T$ be a vector all whose successors are contained in $S \cup T$. Let S_m denote the set of successors of m. Then*

$$p_k(S \cup \{m\} - S_m, n) \leqq 1 + p_k(S \cup T, n).$$

None of the proofs involves any new ideas and we therefore omit them. Theorem 2 now follows by exactly the same argument that led to Theorem 1.

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