A NOTE ON NYMAN'S FUNCTION SYSTEM

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Bertil Nyman [1] proved that the Riemann hypothesis is true if and only if the set

$$\{[\alpha/x] - \alpha[1/x]\}, \qquad 0 < \alpha \leqq 1,$$

is fundamental in $L^2(0, 1)$. We are concerned here with this function system and shall in the sequel denote it by $\{f_\alpha(x)\}$.

1. **The graph of $f_\alpha(x)$.** Every function of the system $\{f_\alpha(x)\}$ has the following properties:

(a) For a fixed α, the function $f_\alpha(x)$ has jumps at the points of the form $\{1/(n+1)\}$ and $\{\alpha/n\}$, where $n=1, 2, 3, \cdots$. These points of discontinuity form a partition of the unit interval and on the resulting set of subintervals of the unit interval the function $f_\alpha(x)$ is constant, respectively.

(b) $|f_\alpha(x)| < 1$ for all $f_\alpha(x)$ of $\{f_\alpha(x)\}$ and $0 \leqq x \leqq 1$.

(c) For rational values of the parameter α there occurs a certain periodicity pattern; in view of its simplicity, we shall not elaborate on it.

2. **The evaluation of $\int_0^1 f_\alpha(x)dx$ and the integrals $\int_0^1 x^m f_\alpha(x)dx$, where $m=1, 2, \cdots, n$.** Introducing the auxiliary function $y=\alpha/x$, consider the expression

$$-(\alpha/x - [\alpha/x]) + \alpha(1/x - [1/x]) \text{ on the unit interval.}$$

Since

$$\int_{\alpha/(j+1)}^{\alpha/j} (\alpha/x - [\alpha/x])dx = \alpha \int_{1/(j+1)}^{1/j} (1/x - [1/x])dx \qquad \text{for } j = 1, 2, 3, \cdots,$$

it follows that

$$\int_0^1 f_\alpha(x)dx = -\int_\alpha^1 \alpha/x dx = \alpha \ln \alpha.$$

Now we set

$$I(m) = \int_0^1 x^m(1/x - [1/x])dx.$$

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Since

$$\int_{\alpha/(j+1)}^{\alpha/j} x^m(\alpha/x - [\alpha/x])dx = \alpha^{m+1}\int_{1/(j+1)}^{1/j} x^m(1/x - [1/x])dx,$$

we obtain

$$\int_0^1 x^m f_\alpha(x)dx = (\alpha - \alpha^{m+1})I(m) - \alpha\int_\alpha^1 x^{m-1}dx$$
$$= (\alpha - \alpha^{m+1})(I(m) - 1/m).$$

We shall denote

$$\int_0^1 x^m f_\alpha(x)dx = F(\alpha, m),$$

and

$$\int_0^1 f_\alpha(x)dx = F(\alpha, 0),$$

and

$$I(m) - 1/m = K(m).$$

Evidently, $K(m) < 0$ for $m = 1, 2, \cdots, n$.

3. The inner product $\int_0^1 P(x)f_\alpha(x)dx$, where $P(x)$ is a polynomial. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with real coefficients. Then

$$\int_0^1 P(x)f_\alpha(x)dx = \sum_{k=0}^n a_kF(\alpha, k).$$

We assume that $\int_0^1 P(x)f_\alpha(x)dx = 0$ for all $0 < \alpha \leqq 1$, but that $\int_0^1 (P(x))^2dx \neq 0$ which means that $P(x) \not\equiv 0$. Thus

$$\sum_{k=0}^n a_kF(\alpha, k) \equiv 0 \qquad \text{for } \alpha \in (0, 1].$$

Writing out the last expression we have

$$a_0\alpha \ln\alpha \equiv -(a_1K(1) + a_2K(2) + \cdots + a_nK(n))\alpha$$
$$+ a_1K(1)\alpha^2 + a_2K(2)\alpha^3 + \cdots + a_nK(n)\alpha^{n+1}$$

for all $\alpha \in (0, 1]$. If $a_0 = 0$, then the coefficients of the polynomial expression above must vanish; since $K(m) < 0$ for $m = 1, 2, \cdots, n$, it follows that $a_1 = a_2 = \cdots = a_n = 0$ also. It can not be, however, that $a_0 \neq 0$ since then we would have an identity relation between a transcendental and an algebraic function. Thus we reached a contradiction and $P(x)\equiv 0$ which means that $\int_0^1 (P(x))^2 dx = 0$.

Reference

1. B. Nyman, *On the one-dimensional translation group and semi-group in certain function spaces*, Dissertation, Univ. of Uppsala, 1950. Math. Rev. **12** (1951), 108; Zbl. Math. **37** (1951), 354.

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A NOTE ON THE LIPSCHITZ CONDITION IN METRIC SPACES

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Let $f\colon M_1 \to M_2$ be a single valued transformation where M_1 and M_2 are metric spaces with metrics d_1 and d_2 respectively.

Definition. $f\colon M_1\to M_2$ is termed Lipschitzian if and only if there exists a finite constant K such that $d_2(f(a), f(b)) \leqq K d_1(a, b)$ for all $a, b\in M_1$.

Theorem. *Let $\{M_i, d_i\}$ be an infinite sequence of metric spaces and $f_{i,j,k}\colon M_i\to M_j$ continuous for all positive integers i, j and k. Then for each i there exists a metric d_i^* for M_i which is equivalent to d_i such that $f_{i,j,k}\colon M_i\to M_j$ is Lipschitzian relative to d_i^* and d_j^* for all i, j and k.*

Proof. Without loss of generality we may assume that each d_i is bounded by 1 on M_i. Let $a, b\in M_s$ and define

$$d_s^*(a, b) \equiv d_s(a, b) + \sum_{q=1}^{\infty} 1/2^{q+n_1+\cdots+n_q+k_1+\cdots+k_q} \cdot d_{n_1}(f_{n_2,n_1,k_1}\cdots f_{n_q,n_{q-1},k_{q-1}} f_{s,n_q,k_q}(a), \cdots (b)),$$

where the summation runs over all $n_1, \cdots, n_q \in S_q$ and $k_1, \cdots, k_q \in S_q$, S_q being the set of all finite sequences of positive integers with q elements; juxtaposition of the f's of course means composition. Then d_s^* satisfies all of the axioms for a metric on M_s. The proof will be complete when we show that (1) d_s^* is equivalent to d_s on M_s and (2) $f_{i,j,k}\colon M_i\to M_j$ is Lipschitzian relative to d_i^* and d_j^*. To show (1) let $y\in M_s$ and $\{y_n\}$ be an infinite sequence in M_s. Suppose $\lim_n d_s^*(y, y_n) = 0$. Then $\lim_n d_s(y, y_n) = 0$ since $d_s(y, y_n) \leqq d_s^*(y, y_n)$. Conversely suppose $\lim_n d_s(y, y_n) = 0$. We note first that

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