ON A CLASS OF QUADRATIC ALGEBRAS

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In a recent paper [3] concerning cubic forms which permit a new type of composition, a class of nonassociative algebras $A$ with 1 over $F$ of characteristic $\neq 2$ is encountered: central simple flexible quadratic algebras satisfying the identity

$$[x, [x, y]] = 0$$

for all $x, y$ in $A$.

We shall denote this class by $\mathcal{C}$. The commutative algebras in $\mathcal{C}$ are the well-known central simple Jordan algebras of degree two. In [3] an example is given (over $F$ containing $\sqrt{-1}$ of a 7-dimensional algebra in $\mathcal{C}$ which is not commutative. We shall denote by $\mathcal{C}'$ the class of all algebras in $\mathcal{C}$ which are not commutative.

In this note we show that, in the presence of the other conditions on the algebras in $\mathcal{C}$, identity (1) may be replaced by

$$[x, y]^2 = 0$$

for all $x, y$ in $A$.

For arbitrary $F$ of characteristic $\neq 2$ and for any dimension $\geq 7$ (possibly infinite), we show that $\mathcal{C}'$ contains an algebra $A$ of dimension $n$ over $F$. On the other hand, if $F$ is also of characteristic $\neq 3$, then $\mathcal{C}'$ contains no algebra of dimension $\leq 6$.

An algebra $A$ (of possibly infinite dimension) with 1 over a field $F$ is called a quadratic algebra in case $A \neq F1$ and, for every $x$ in $A$, there are $t(x), n(x)$ in $F$ such that

$$x^2 - t(x)x + n(x)1 = 0.$$ 

It is well-known that, by defining $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$, the trace $t(x)$ is linear and the norm $n(x)$ is a quadratic form. If $F$ has characteristic $\neq 2$, there is a (possibly infinite) basis $U = \{u_i\}$ of $A$ over $F$ such that $1 = u_0 \in U$,

$$u_i^2 = \beta_i 1,$$

$$\beta_i \in F, \beta_0 = 1,$$

and, for $i \neq 0, j \neq 0, i \neq j$,

$$u_i u_j = \sum \pi_{ijk} u_k = - u_j u_i$$

where only a finite number of the $\pi_{ijk}$ are $\neq 0$. From (5) we have

$$\pi_{ijk} = - \pi_{jik}$$

for all $k$.

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Any $x \in A$ may be written uniquely in the form

\[ x = \sum \alpha_i u_i \]

with only a finite number of the $\alpha_i \neq 0$. Then

\[ t(x) = 2\alpha_0, \quad n(x) = \alpha_0 - \sum_{i \neq 0} \alpha_i^2 \beta_i. \]

It is well-known that $n(x)$ is nondegenerate if and only if in (4) we have $\beta_i \neq 0$ for every $i$. One sees easily [2; 3] that $A$ is central simple if and only if $n(x)$ is nondegenerate and the dimension of $A$ is $\geq 3$. Clearly $A$ is commutative if and only if all $\pi_{ijk} = 0$ in (5).

An algebra $A$ over $F$ is called flexible in case

\[ (xy)x = x(yx) \]

for all $x, y \in A$.

It is shown in [1, p. 588] that a quadratic algebra is flexible if and only if

\[ \pi_{ij0} = \pi_{ij1} = \pi_{ij2} = 0 \quad (i \neq j; i \neq 0, j \neq 0) \]

and

\[ \beta_i \pi_{jkl} = \beta_k \pi_{ijkl} \quad (i, j, k \text{ distinct}; i \neq 0, j \neq 0, k \neq 0) \]

are satisfied in (5). Hence the algebras in $\mathcal{C}$ are the algebras $A$ of dimension $\geq 3$ over $F$ of characteristic $\neq 2$ satisfying (4) with all $\beta_i \neq 0$, (5), (6), (10), (11), and (1) where $[x, y]$ is the commutator $[x, y] = xy - yx$. The algebras in $\mathcal{C}'$ satisfy the additional requirement that at least one $\pi_{ijk}$ in (5) is $\neq 0$.

Let $(x, y)$ be the nondegenerate symmetric bilinear form such that $n(x) = (x, x)$. In [3, equation (78)] it is shown that, for any $A$ in $\mathcal{C}$, we have

\[ n(xy) = (xy, yx) = n(yx) \]

for all $x, y$ in $A$.

Hence $n([x, y]) = (xy - yx, xy - yx) = n(xy) - 2(xy, yx) + n(yx) = 0$ for all $x, y$ in $A$. Since also $l([x, y]) = 0$ [3, p. 172], we see that (3) implies (2). But then we may replace (1) by (2) in the definition of $\mathcal{C}$. For, conversely, we may linearize (2) in $y$ to obtain

\[ [x, y][x, z] + [x, z][x, y] = 0. \]

In [3, p. 172] it is shown that flexibility implies

\[ l(xy) = l(yx), \quad l((xy)z) = l(x(yz)), \]

and
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\[ (x, y) = \frac{1}{2} t(x y) \quad \text{where} \quad \tilde{y} = t(y)1 - y. \]

Hence (12) implies

\[ 0 = t([x, y][x, z] + [x, z][x, y]) = 2t([x, y] [x, z]) = 2t((xy)(xz) - (zx)(xy) - (yx)(xz) + (zx)(yx)) = 2t((xyx)z - z(x(xy)) - (yx)xz + z(xyx)) = (2xyx - x(xy) - (yx)x, z) \]

for every \( z \) in \( A \). Since \( (x, y) \) is nondegenerate, we have \( 2xyx - x(xy) - (yx)x = 0 \), implying (1).

Although (1) has the advantage of being an identity of lower degree than identity (2), the latter may be very easily stated: the square of every commutator is 0. We shall use this in the proof of the Theorem below.

**Lemma.** Let \( A_1 \) over \( F \) be an algebra in \( \mathcal{C}' \), and let the dimension of \( A_1 \) be \( m \). Then, for any \( n \geq m \) (possibly infinite), there exists an \( n \)-dimensional algebra \( A \) over \( F \) in \( \mathcal{C}' \) such that \( A_1 \) is a subalgebra of \( A \).

**Proof.** A basis \( U_1 \) for \( A_1 \) (of the special form we have chosen) may be extended to a basis \( U = U_1 \cup U_2 \) for an algebra \( A \) over \( F \) such that \( A \) has dimension \( n \) over \( F \). It remains to define multiplication appropriately for pairs of elements in \( U \) where at least one of the elements of the pair is in \( U_2 \). Define

\[ u_i^2 = -1 \quad \text{for} \quad u_i \in U_1, \]

and

\[ u_i u_j = 0 \quad \text{for} \quad i \neq j, \text{if either} \ u_i \text{or} \ u_j \text{is in} \ U_2. \]

All of the conditions for \( A \) to be in \( \mathcal{C}' \) are obviously satisfied, except possibly for (11) and (2) which we verify as follows. If \( u_i, u_j, \) and \( u_k \) are all in \( U_1 \), then (11) is satisfied since \( A_1 \) is in \( \mathcal{C} \). If both \( u_i \) and \( u_j \) are in \( U_1 \), then \( u_i u_j \) in \( A_1 \) implies \( \pi_{ijk} = 0 \) for all \( k \) for which \( u_k \in U_2 \); hence (14) implies (11) in this case. Finally, if either \( u_i \) or \( u_j \) is in \( U_2 \), then \( \pi_{ijk} = \pi_{jki} = 0 \) by (14). To verify (2), we write \( x \) in \( A \) in the form \( x = x_1 + x_2 \) where \( x_1 \in A_1 \) and where \( x_2 \) is a linear combination of elements of \( U_2 \). With \( y = y_1 + y_2 \) written similarly, we obtain \( [x, y] = [x_1, y_1] \) by (14). Then \( [x, y]^2 = [x_1, y_1]^2 = 0 \) since \( A_1 \) is in \( \mathcal{C} \).

**Theorem.** Let \( F \) be an arbitrary field of characteristic \( \neq 2 \). For any dimension \( n \geq 1 \) (possibly infinite), there exists an \( n \)-dimensional algebra
A over $F$ in the class $G'$. If $F$ is also of characteristic $\neq 3$, there are no algebras $A$ of dimension $\leq 6$ over $F$ in $G'$.

PROOF. We use the lemma to see that (i) the first statement in the conclusion of the Theorem may be established by constructing a 7-dimensional algebra in $G'$, while (ii) the final statement may be proved by showing that any 6-dimensional algebra in $G$ is commutative (assuming characteristic $\neq 3$).

To construct an example in (i), let $1, u_1, u_2, \ldots, u_6$ be a basis for $A$ over $F$. Define

$$u_i = 1, \quad u_{i+3} = 1, \quad u_i u_{i+3} = u_{i+2} u_i = 0 \quad (i = 1, 2, 3).$$

For cyclic permutations $i, j, k$ of 1, 2, 3, define

$$u_i u_j = u_i u_{j+3} = u_{i+3} u_j = u_{i+3} u_{j+3} = u_k - u_{k+3} = -u_j u_i = -u_{j+3} u_{i+3} = -u_{j+3} u_{i+3}.$$

Then (4) with all $\beta_i \neq 0$, (5), (6), (10) and (11) are satisfied. In order to verify (1) we note that, relative to the basis $1, u_1, u_2, \ldots, u_6$, the matrix $T_x$ of the linear transformation $y \rightarrow [x, y]/2$ has the form

$$T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & -K \\ 0 & K & -K \end{pmatrix},$$

where

$$K = \begin{pmatrix} 0 & -\alpha_3 - \alpha_5 & \alpha_2 + \alpha_6 \\ \alpha_5 + \alpha_6 & 0 & -\alpha_1 - \alpha_2 \\ -\alpha_2 - \alpha_6 & \alpha_1 + \alpha_4 & 0 \end{pmatrix}$$

for $x$ in (7).

Hence $T_x^2 = 0$, implying (1).

To prove (ii), we may as well assume that $F$ is algebraically closed. For, if $\Sigma$ is the algebraic closure of $F$, and if $A$ over $F$ is in the class $G'$, then $A_\Sigma$ is in the corresponding class of algebras over $\Sigma$ (the equations (4), (5), (6), (10), and (11) remain unchanged, and one may linearize (1) to obtain an equivalent multilinear identity). Hence we may assume that $\beta_i = -1$ ($i = 1, \ldots, 5$) in (4). The 60 elements $\pi_{ijk}$ which are not given as 0 by (10) are partitioned by the relations (6) and (11) into 10 classes, within each of which the elements differ only by a factor of $\pm 1$. Writing subscripts modulo 5 (using 5, not 0), we make the following abbreviations for representatives of these 10 classes:
Then, for subscripts modulo 5, we have

\[ u_i u_{i+1} = \sigma_{i+4} u_{i+2} + \epsilon_{i+2} u_{i+3} + \sigma_{i+3} u_{i+4} \quad (i = 1, \ldots, 5) \]

and

\[ u_i u_{i+2} = -\epsilon_{i+2} u_{i+1} - \epsilon_{i+4} u_{i+3} + \sigma_{i+3} u_{i+4} \quad (i = 1, \ldots, 5). \]

Now (2), (5) and (4) imply \([u_i, u_j]^2 = (2u_i u_j)^2 = -4\Sigma x_{ij} = 0\) for \(i \neq j\). Hence (16) implies

\[ \frac{\sigma_{i+4} + \epsilon_{i+2} + \sigma_{i+3}^2}{2} = 0 \quad (i = 1, \ldots, 5), \]

and (17) implies

\[ \frac{\epsilon_{i+2} + \epsilon_{i+4} + \sigma_{i+3}^2}{2} = 0 \quad (i = 1, \ldots, 5). \]

Eliminating the \(\epsilon\)'s in (18) and (19), and adding the resulting equations, we arrive easily at \(\sigma_i = \epsilon_i = 0\) \((i = 1, \ldots, 5)\) since the characteristic of \(F\) is \(\neq 2, 3\). Hence \(A\) is commutative.

Remark. Characteristic \(\neq 3\) is not required in a direct proof that there are no algebras in \(\mathfrak{C}'\) of dimension \(\leq 5\).

References


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