A NOTE ON TORSION FREE ABELIAN GROUPS OF INFINITE RANK

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1. **Introduction.** R. A. Beaumont and R. S. Pierce [1] have constructed a set of invariants for the quasi-isomorphism classes of quotient-divisible torsion free groups of finite rank. (See §3 for a definition of quotient-divisible groups. We will not be concerned with quasi-isomorphism here.) In a future publication the author hopes to show that these invariants can be extended to quotient-divisible groups of arbitrary rank. Hence it is of interest to know which torsion free groups are quotient-divisible. The object of this note is to prove Theorem 2.1 which enables us to shed some light on the above problem and to settle completely the related problem of determining which torsion free groups can be written as the sum (not necessarily direct) of two free subgroups.

All groups under consideration are abelian.

2. **The main theorem.**

THEOREM 2.1. *Let $\{A_\nu | \nu \in N\}$ and $\{B_\nu | \nu \in N\}$ be families of subgroups of the torsion free group G and put $B^* = \bigcap_{\nu \in N} B_\nu$. Assume that*

(i) *$A_\nu \subseteq B_\nu$ for each $\nu \in N$,*

(ii) *$B^* + A_\nu = B_\nu$ for each $\nu \in N$,*

(iii) *for each $\nu \in N$ there exists a set $\{x(i, \nu) | (i, \nu) \in I(\nu)\}$ of generators of B_ν* mod *A_ν, contained in B^* and such that, for each $\mu \in N$, $\sum_{\nu \in N} |I(\nu)| \leq$ rank $B^* \cap A_\mu$.*

Then there exists a free subgroup F of G such that $F + A_\nu = B_\nu$ for each $\nu \in N$.

PROOF. Put $K = \bigcup_{\nu \in N} I(\nu)$ and let α be the least ordinal number such that $|\alpha| = |K|$. Let ϕ be a one-to-one mapping of $\{\beta | \beta < \alpha\}$ onto K and let ρ be the mapping of $\{\beta | \beta < \alpha\}$ onto N given by $\rho(\beta) = \nu$ where $\phi(\beta) \in I(\nu)$. (The $I(\nu)$ are mutually disjoint.) We define by transfinite induction a set $\{y_\beta | \beta < \alpha\}$ satisfying

(a) $\{y_\beta | \beta < \alpha\} \subseteq B^*$,

(b) $\{y_\beta | \beta < \alpha\}$ is independent,

(c) for each $\beta < \alpha$, $y_\beta \equiv x(\phi(\beta))$ mod $A_{\rho(\beta)}$.

Put $y_0 = x(\phi(0))$.

Assume that for some $\gamma < \alpha$, y_β has been chosen for all $\beta < \gamma$ such

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that $\{y_\beta | \beta<\gamma\}$ satisfies (a), (b) and (c) above. Let C be the pure subgroup of B^* generated by $\{y_\beta | \beta<\gamma\}$. Then rank $C=|\gamma|<|\alpha| =|K| = \sum_{\nu\in N} |I(\nu)| \leqq \text{rank } B^*\cap A_{\rho(\gamma)}$, so that $B^*\cap A_{\rho(\gamma)} \not\subseteq C$. Choose $a_\gamma \in B^*\cap A_{\rho(\gamma)}$ such that $a_\gamma \notin C$. Now put

$$y_\gamma = \begin{cases} x(\phi(\gamma)) & \text{if } x(\phi(\gamma)) \notin C, \\ x(\phi(\gamma)) + a_\gamma & \text{if } x(\phi(\gamma)) \in C. \end{cases}$$

Then $y_\gamma \notin C$ so that $\{y_\beta | \beta \leqq \gamma\}$ is independent. However, $y_\gamma \in B^*$ since $x(\phi(\gamma))$ and a_γ are in B^* so that $\{y_\beta | \beta\leqq\gamma\} \subseteq B^*$. Finally it is clear that $\{y_\beta | \beta\leqq\gamma\}$ satisfies (c). We thus obtain for each $\beta<\alpha$ an element y_β such that $\{y_\beta | \beta<\alpha\}$ satisfies (a), (b) and (c) above.

Let F be the subgroup of G generated $\{y_\beta | \beta<\alpha\}$. Then F is free by (b) and $F\subseteq B^*\subseteq B_\nu$ for each ν by (a). Hence $F+A_\nu\subseteq B_\nu$ for each ν. By (c) however we have for each $(i, \nu)\in I(\nu)$, $y_{\phi^{-1}(i,\nu)} \equiv x(i, \nu) \bmod A_\nu$ so that by hypothesis (iii), $F+A_\nu \supseteq B_\nu$. Thus $F+A_\nu = B_\nu$ as required.

THEOREM 2.2. *The torsion free group G can be written as the sum of two free subgroups if and only if G is free or G has infinite rank.*

PROOF. If G is free the theorem is trivial. Suppose then that G has infinite rank and let A be any free subgroup of G such that rank A = rank G. Then $|G/A| \leqq |G|$ = rank G = rank A. Taking $\{A_\nu | \nu\in N\} = \{A\}$ and $\{B_\nu | \nu\in N\} = \{G\}$ in Theorem 2.1 we obtain a free subgroup F of G such that $F+A=G$.

Conversely, if $G=A+F$ with A and F free and rank G finite then F has a finite number of generators and we can assume (by expanding A if necessary) that G/A is a torsion group. Then $G/A \cong F/A\cap F$ is a finitely generated torsion group and is therefore finite. Thus, for some integer $n\neq 0$, we have $nG\subseteq A$ from which it follows that nG, and hence G, is free.

One can obtain a result similar to Theorem 2.2 for mixed groups. For this purpose we say that a mixed group M is *almost torsion* if $M=F\oplus T$ with F free and T the maximal torsion subgroup of M. By the torsion free rank of a mixed group M we mean the rank of M/T where T is again the maximal torsion subgroup of M.

THEOREM 2.3. *Let G be a mixed group of infinite torsion free rank. Then G is the sum of two almost torsion groups A and B whose intersection is an almost torsion group with the same maximal torsion subgroup as G. A mixed group G of finite torsion free rank is the sum of two almost torsion subgroups, each containing the maximal torsion subgroup of G, if and only if G is almost torsion.*

PROOF. Let T be the maximal torsion subgroup of G and suppose that G/T has infinite rank. Then by Theorem 2.2 there exist subgroups H and K of G such that $T\subseteq H\cap K$, H/T and K/T are free and $G/T=H/T+K/T$. Since H/T and K/T are free, H and K are almost torsion. Clearly $G=H+K$. Finally, since $H\cap K/T\subseteq H/T$, $(H\cap K)/T$ is free and it follows that $H\cap K$ is almost torsion with maximal torsion subgroup T.

Now suppose rank G/T is finite, $G=H+K$ with H and K almost torsion and $T\subseteq H\cap K$. Then $G/T=H/T+K/T$ is the sum of two free groups. By Theorem 2.2, G/T is free so that G is almost torsion. The converse is trivial.

3. **Quotient-divisible groups.**[1] Following Beaumont and Pierce [1], we make

DEFINITION 3.1. A torsion free group G is *quotient-divisible* if G/F is a divisible torsion group for some free subgroup F of G.

In the sequel we denote by $Z(S)$ the subgroup of G generated by the subset S of G.

THEOREM 3.2. *A torsion free group G of finite rank which is divisible by almost all primes is quotient divisible.*

PROOF. For each prime p, G/pG is a direct sum of cyclic groups of order p, the number of summands being less than or equal to rank G. Thus, if G has finite rank and $pG=G$ for almost all primes p, there exists a finite set X of elements of G such that $Z(X)+pG=G$ for all primes p. Since X is finite and G is torsion free, $Z(X)$ is free and the equations $Z(X)+pG=G$ yield $p(G/Z(X))=G/Z(X)$ for all primes p. Thus, $G/Z(X)$ is divisible.

LEMMA 3.3. *A torsion free group G is quotient divisible if and only if there exists an independent set S in G such that $Z(S)+pG=G$ for all primes p.*

PROOF. This is little more than a restatement of the definition since the group G/H is divisible if and only if $p(G/H)=G/H$ for all primes p, a condition which is equivalent in turn to $H+pG=G$ for all p. The subgroup H is free if and only if $H=Z(S)$ for some independent set S in G and by expanding S to a maximal independent set of G we can ensure that $G/Z(S)$ is torsion.

THEOREM 3.4. *Let G be a torsion free group, r the rank of G and $r(p)$ the rank of G/pG for each prime p (i.e. $r(p)$ is the dimension of G/pG considered as a vector space over the prime field of characteristic p). If*

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[1] The results of §3 appeared in the author's Ph.D. thesis (University of Washington, 1960). The author would like to express his appreciation of several fruitful conversations with Professor R. S. Pierce on the ideas covered in this paper.

$r \geqq \sum_p r(p)$ then G is quotient divisible.

PROOF. For each p, G/pG is a direct sum of cyclic groups of order p: $G/pG = \sum_{(i,p)\in I(p)} Z(x(i, p)+pG)$ for some elements $x(i, p)\in G$. Evidently, $r(p)=|I(p)|$. Since rank $pG=$rank $G=r$ for each p, we may apply Theorem 2.1 with $N=\{p \mid p \text{ is a prime}\}$, $A_p=pG$ for each $p\in N$ and $B_p=G$ for each $p\in N$, and obtain a free group $F\subseteq G$ such that $F+pG=G$ for each p. Applying Lemma 3.3, with S any basis of F, we conclude that G is quotient divisible.

COROLLARY 3.5. *Any torsion free group of infinite rank is quotient divisible.*

PROOF. $\sum_p r(p) \leqq \sum_p r = \aleph_0 r = r$.

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CONVOLUTIONS OF SLOWLY OSCILLATING FUNCTIONS

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1. **Introduction.** In the study of asymptotic formulae for arithmetic functions we invariably come upon such functions as $x^\alpha \log^\beta x$, li x, $x^\alpha(\log x)^\beta(\log\log x)^\gamma$, $x^\alpha \exp(\beta \log^{\gamma_0} x)$. If α is complex and β and γ are real numbers ($\gamma_0<1$) then these functions are of the form $x^\alpha L(x)$, where L is a *slowly oscillating* function; i.e., a continuous positive valued function on $[x_0, \infty)$ for some $x_0\geqq 1$ such that

$$\lim_{x\to\infty} L(cx)/L(x) = 1 \tag{1}$$

for each $c>0$. A common approach to asymptotic formulae is the convolution method of Landau which was formulated into a general theorem by the author [**4**]. The resulting main terms involve convolutions of functions $x^\alpha L(x)$. In the present paper we shall give conditions under which such convolutions are also in the form $x^\alpha L(x)$ or nearly so.

Now it is known [**1**; **3**] (see [**2**] for other properties) that a function L on $[x_0, \infty)$ is slowly oscillating if and only if there exist continuous functions ρ and δ on $[x_0, \infty)$ such that $\rho(x)>0$, $\rho(x)\to\rho_0>0$, $\delta(x)\to 0$ as $x\to\infty$ and

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