\[ r \geq \sum_p r(p) \text{ then } G \text{ is quotient divisible.} \]

**Proof.** For each \( p \), \( G/pG \) is a direct sum of cyclic groups of order \( \varphi(p) \), \( G/pG = \sum_{\{i, p\} \in \{p\}} Z(x(i, p) + pG) \) for some elements \( x(i, p) \in G \). Evidently, \( r(p) = |I(p)| \). Since rank \( pG = \text{rank } G = r \) for each \( p \), we may apply Theorem 2.1 with \( N = \{ p \mid p \text{ is a prime} \} \), \( A_p = pG \) for each \( p \in N \) and \( B_p = G \) for each \( p \in N \), and obtain a free group \( F \subseteq G \) such that \( F + pG = G \) for each \( p \). Applying Lemma 3.3, with \( S \) any basis of \( F \), we conclude that \( G \) is quotient divisible.

**Corollary 3.5.** Any torsion free group of infinite rank is quotient divisible.

**Proof.** \( \sum_p r(p) \leq \sum_p r = \aleph_0 \sigma = r. \)

**Bibliography**


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**CONVOLUTIONS OF SLOWLY OSCILLATING FUNCTIONS**

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1. **Introduction.** In the study of asymptotic formulae for arithmetic functions we invariably come upon such functions as \( x^\alpha \log^\beta x \), \( \text{li } x \), \( x^\alpha (\log x)^\gamma (\log \log x)^\delta \), \( x^\alpha \exp(\beta \log^\gamma x) \). If \( \alpha \) is complex and \( \beta \) and \( \gamma \) are real numbers \((\gamma_0 < 1)\) then these functions are of the form \( x^\alpha L(x) \), where \( L \) is a slowly oscillating function; i.e., a continuous positive valued function on \([x_0, \infty)\) for some \( x_0 \geq 1 \) such that

\[
(1) \quad \lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1
\]

for each \( c > 0 \). A common approach to asymptotic formulae is the convolution method of Landau which was formulated into a general theorem by the author [4]. The resulting main terms involve convolutions of functions \( x^\alpha L(x) \). In the present paper we shall give conditions under which such convolutions are also in the form \( x^\alpha L(x) \) or nearly so.

Now it is known [1; 3] (see [2] for other properties) that a function \( L \) on \([x_0, \infty)\) is slowly oscillating if and only if there exist continuous functions \( \rho \) and \( \delta \) on \([x_0, \infty)\) such that \( \rho(x) > 0, \rho(x) \to \rho_0 > 0, \delta(x) \to 0 \) as \( x \to \infty \) and

\[
\delta(x) \to 0 \text{ as } x \to \infty
\]

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We shall be concerned in particular with the special slowly oscillating functions having the form

\[ L(x) = \rho(x) \exp \int_{-\infty}^{x} t^{-\delta(t)} dt. \]

where \( \rho > 0 \) is constant and \( \delta \) is a bounded measurable function with \( \delta(x) = o(1) \) as \( x \to \infty \). By a convolution of two functions \( A \) and \( B \) we shall mean one of

\[ \int_{a}^{x/a} A(x/u) B(u) \, du \quad \text{or} \quad \int_{a}^{x/a} \frac{k(u) A(x/u) B(u)}{u^{\gamma}} \, du. \]

The former of these expressions is called the Stieltjes resultant of \( A \) by \( B \) on \([a^2, \infty)\).

2. A convolution theorem. Suppose \( \alpha \) and \( \beta \) are real numbers with \( \beta \neq 0 \) and that \( L \) and \( M \) are slowly oscillating with \( M \) special. Then for sufficiently large \( a \geq 1 \) the Stieltjes resultant on \([a^2, \infty)\) of \( x^\alpha L(x) \) by \( x^\beta M(x) \) has the form \( \gamma x^\gamma N(x) \) where \( \gamma = \max(\alpha, \beta) \) and \( N(x) \) is slowly oscillating for \( x > a^2 \). Further, \( N(x) \) is asymptotically proportional to \( L(x) \) if \( \alpha > \beta \) and to \( M(x) \) if \( \alpha < \beta \).

Proof. If \( M \) is special then we have

\[ M(x) = \rho(x) \exp \int_{1}^{x} t^{-\delta(t)} dt \]

with \( \rho > 0 \), \( \delta(t) = o(1) \) as \( t \to \infty \). Choose \( a \geq 1 \) so that \( |\delta(t)| < |\beta| \) for all \( t \geq a \) and thus \( \beta + \delta(t) \) remains constant in sign for \( t \geq a \). Now the Stieltjes resultant in question is

\[ \int_{a}^{x/a} (x/u)^{\alpha} L(x/u) d(\omega^\alpha M(u)) \]

\[ = \int_{a}^{x/a} (x/u)^{\alpha} L(x/u) u^{\beta-1} M(u) (\beta + \delta(u)) du \]

\[ = x^\alpha \int_{a}^{x/a} u^{-(\alpha-\beta)-1} L(x/u) M(u) (\beta + \delta(u)) du \]

\[ = x^\beta \int_{a}^{x/a} u^{-(\beta-\alpha)-1} M(x/u) (\beta + \delta(x/u)) L(u) du. \]
CONVOLUTIONS OF SLOWLY OSCILLATING FUNCTIONS

If \( \alpha > \beta \) we apply Lemma 1 below to (1) with
\[
A(x) = x^{-(\alpha-\beta)-1}M(x)(\beta + \delta(x)) = O(x^{-(\alpha-\beta)-1+\epsilon})
\]
for each \( \epsilon > 0 \) (e.g., \( \epsilon = (\alpha-\beta)/2 \)). Thus the resultant is asymptotic to
\[
x^a L(x) \int_{a}^{\infty} u^{-(\alpha-\beta)-1} M(u)(\beta + \delta(u)) \, du
\]
as \( x \to \infty \), if \( \alpha > \beta \). (The integral is different from 0 by our choice of \( a \).)

Similarly, if \( \beta > \alpha \), we apply Lemma 1 to (2) and find the resultant asymptotic to
\[
\beta x^a M(x) \int_{a}^{\infty} u^{-(\beta-\alpha)-1} L(u) \, du.
\]

If \( \alpha = \beta \), (1) gives
\[
x^a \int_{a}^{z/a} u^{-1} L(x/u) M(u)(\beta + \delta(u)) \, du,
\]
to which we apply Lemma 2.

3. Two lemmas. **Lemma 1.** Suppose \( A \) and \( B \) are measurable functions on \([a, \infty)\) \((a > 0)\) with
\[
A(x) = O(x^{-\kappa})
\]
for some \( \kappa > 1 \). Suppose \( B \) is positive valued, bounded on each bounded interval and asymptotic to a slowly oscillating function. Then
(1)
\[
\lim_{z \to \infty} \int_{a}^{z/a} A(u)(B(x/u)/B(x)) \, du = \int_{a}^{\infty} A(u) \, du.
\]

**Proof.** Set \( B(x) = 0 \) for \( 0 \leq x < a \) and apply the Lebesgue dominated convergence theorem to
\[
\int_{a}^{z/a} A(u)(B(x/u)/B(x)) \, du = \int_{a}^{\infty} A(u)(B(x/u)/B(x)) \, du.
\]
Now for all \( u \geq a \)
(2)
\[
\lim_{x \to \infty} A(u)B(x/u)/B(x) = A(u).
\]
With the aid of (1.2), for each \( \epsilon > 0 \)
\[
B(x/u)/B(x) = O(u^\epsilon)
\]
uniformly for \( u \geq a, x \geq K \) (sufficiently large), and so we have
\[ A(u)B(x/u)/B(x) = O(u^{-\varepsilon}) \]

with \(0 < \varepsilon < \kappa - 1\). Thus since \(u^{-\varepsilon}\) is absolutely integrable over \([a, \infty)\), (2) and (3) imply (1).

**Lemma 2.** Suppose \(L\) is slowly oscillating and \(B\) is measurable and asymptotic to a slowly oscillating function. Then for sufficiently large \(a\)

\[ N(x) = \int_a^x u^{-1}L(x/u)B(u)du = \int_a^x u^{-1}L(u)B(x/u)du \]

is slowly oscillating for \(x \geq x_0 > a^2\).

The proof is elementary. We use the fact that \(\int_a^x u^{-1}L(u)du\) is slowly oscillating and that

\[ \lim_{x \to \infty} \frac{B(cx)}{B(x)} = 1 \]

uniformly for \(k_1 \leq c \leq k_2\) for each \(k_2 > k_1 > 0\).

4. **Remarks.** If \(\alpha = \beta = 0\) we have the Stieltjes resultant of \(L\) by \(M\). We can show it slowly oscillating under various hypotheses. In particular if \(L\) and \(M\) are both special and nondecreasing (\(M\) nonconstant).

Note that \(A\) in Lemma 1 need not be real-valued. Thus if \(\alpha\) and \(\beta\) are complex with \(\Im \alpha \neq \Im \beta\) the Stieltjes resultant of \(x^\alpha L(x)\) by \(x^\beta M(x)\) is \(x^\gamma N(x)\) with \(N(x)\) complex-valued but asymptotically proportional to \(L(x)\) or \(M(x)\) according as \(\Re \alpha > \Re \beta\) or \(\Re \beta > \Re \alpha\). \(\gamma\) would be \(\alpha\) in the former case and \(\beta\) in the latter.

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