ON THE ABSOLUTE CESÀRO SUMMABILITY OF FOURIER SERIES

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1. Let \( f(t) \) be an even integrable function with period \( 2\pi \) and let

\[
f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt,
\]

\[
\Phi_n(t) = \int_0^t (t-u)^{n-1}f(u)du, \quad \alpha > 0.
\]

L. S. Bosanquet [2] proved the following:

**Theorem A.** If \( \Phi_n(t)/t^n \) is of bounded variation in \( (0, \pi) \), then, for \( \gamma > \alpha \), the Fourier series of \( f(t) \) at \( t=0 \) is evaluable \( (C, \gamma) \), and conversely, if the Fourier series of \( f(t) \) at \( t=0 \) is evaluable \( (C, \gamma) \), then, for \( \alpha > \gamma + 1 \), \( \Phi_n(t)/t^n \) is of bounded variation in \( (0, \pi) \).

Earlier, L. S. Bosanquet [1] proved the following:

**Theorem B.** If \( \Phi_n(t)/t^n \) is of bounded variation in an interval to the right of \( t=0 \), then the Fourier series of \( f(t) \) at \( t=0 \) is evaluable \( (C, \alpha - 1) \) when \( \alpha \geq 1 \).

The purpose of this paper is to establish the following:

**Theorem.** If the Fourier series of \( f(t) \) at \( t=0 \) is evaluable \( (C, \alpha) \) to zero, then \( \Phi_n(t) = o(t^{\alpha}) \) when \( \alpha \geq 1 \).

This theorem was implicitly proved by N. Obreschkoff [4, Satz 2] when \( \alpha = 1 \).

2. Preliminary lemmas.

**Lemma 1.** Let a series \( \sum_{n=1}^{\infty} b_n \) be convergent and let

\[
c_n = \sum_{r=n}^{\infty} b_r.
\]

Then

\[
\sum_{r=n}^{m} a_r b_r = a_n c_n - a_m c_{m+1} - \sum_{r=n}^{m-1} c_{r+1} \Delta a_r,
\]  

(2.1)

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where $\Delta a_r = a_r - a_{r+1}$. In particular, if $a_m c_{m+1} \to 0$ as $m \to \infty$,

\begin{equation}
\sum_{r=n}^{\infty} a_r b_r = a_n c_n - \sum_{r=n}^{\infty} c_{r+1} \Delta a_r.
\end{equation}

Proof is obvious.

Throughout this paper, let

\[ \lfloor x \rfloor \]

be the greatest integer less than $x$, 

\[ \beta = \lfloor \alpha \rfloor, \]

\[ A_n^\gamma = \binom{\gamma + n}{n} = \frac{\Gamma(n + \gamma + 1)}{\Gamma(n + 1) \Gamma(\gamma + 1)}, \]

\[ s_n^\gamma = \sum_{r=0}^{n} A_{n-r}^\gamma a_r, \]

\[ \phi(n, t) = \int_0^t (t^2 - u^2)^{\gamma-1} \cos \pi u \, du, \]

\[ \Delta^k \phi(n, t) = \Delta^{k-1} \phi(n, t) - \Delta^{k-1} \phi(n + 1, t), \quad \Delta^0 \phi(n, t) = \phi(n, t), \]

\[ \psi(\mu, t) = 2^\mu \Gamma(\mu + 1) t^{-\mu} J_{\mu}(t), \]

where $J_{\mu}(t)$ is the Bessel function of order $\mu$, $\mu > -1/2$.

**Lemma 2.** For $\alpha > 0$ and $k = 0, 1, 2, \ldots$,

\begin{equation}
\Delta^k \phi(n, t) = O(n^{-\alpha k + k - 1}).
\end{equation}

**Proof.** Since

\begin{equation}
\psi \left( \alpha - \frac{1}{2}, n t \right) = \frac{2 \Gamma(\alpha + 1/2)}{\Gamma(1/2) \Gamma(\alpha)} t^{1-2\alpha} \phi(n, t),
\end{equation}

we have the lemma by a known result. (See K. Chandrasekharan and O. Szász [3, pp. 716 and 728].)

**Lemma 3.** For $\alpha > 0$ and $j = 0, 1, 2, \ldots$, such that $\alpha + j \geq 1$,

\begin{equation}
\sum_{n=\rho}^{\infty} A_{n-\rho}^\beta \phi(n, t) = O(t^{-2\alpha - 2j}).
\end{equation}

**Proof.** The lemma in which $\alpha$ is an integer is obvious by (2.3), so that we shall prove the lemma in which $\alpha$ is not an integer. Putting $\rho = \lfloor t^{-1} \rfloor$ and using (2.2) and (2.3),
\[
\sum_{n=m}^{\infty} A_n^{\beta-a} \phi(n, t) = \left( \sum_{n=m}^{r-1} + \sum_{n=r+1}^{\infty} \right) \\
= O \left( \sum_{n=m}^{r-1} \sum_{n=m}^{r-1} A_n^{\beta-a} \phi(n + r, t) \right) + A_r^{\beta-a} \phi(n + 1, t) \\
+ \sum_{n=r+1}^{\infty} A_n^{\beta-a} \phi(n + 1, t) \\
= O(\phi(n + 1, t)) + O(\phi(n + 1, t)) + O(\phi(n + 1, t)) \\
= O(\phi(n + 1, t)),
\]
which is the required result.

**Lemma 4.** Let \(\alpha \geq 1\). Then, for any \(m\) and \(t > 0\),

\[(2.6) \quad U_m(t) = \sum_{r=1}^{m} A_r^{\beta-a} \sum_{n=m}^{\infty} A_n^{\beta-a} \phi(n, t) = O(t^{2a-1}).\]

**Proof.** If \(mt \leq 1\), then, using (2.5) for \(j' = 2\),

\[(2.7) \quad U_m(t) = O(\sum_{r=1}^{m} A_r^{\beta-a} \phi(n, t)) = O(t^{2a-1}) = O(t^{2a-1}).\]

Therefore we shall prove (2.6) when \(mt > 1\). Putting \(\rho = [t^{-1}]\), we write

\[U_m(t) = \left( \sum_{r=1}^{m} + \sum_{r=m}^{\infty} \right) = U_{m_1}(t) + U_{m_2}(t),\]

where \(U_{m_1}(t) = O(t^{2a-1})\) by (2.7). Since, for \(j = 0, -1\), using (2.5),

\[
\sum_{r=m}^{\infty} A_n^{\beta-a} \phi(n, t) \\
= \lim_{p \to \infty} \sum_{r=m}^{p} \left( \sum_{n=0}^{\infty} A_n^{\beta-a} \phi(n + r, t) - \sum_{n=0}^{\infty} A_n^{\beta-a} \phi(n + r + 1, t) \right) \\
= \lim_{p \to \infty} \left( \sum_{n=0}^{\infty} A_n^{\beta-a} \phi(n + r, t) - \sum_{n=0}^{\infty} A_n^{\beta-a} \phi(n + r + 1, t) \right) \\
= \sum_{n=0}^{\infty} A_n^{\beta-a} \phi(n, t),
\]
we have, by (2.1) and (2.5),

\[ U_{2n}(t) = A^\alpha \sum_{n=0}^\infty A_{n,p} \Delta^{\beta+1} \phi(n, t) - A^\alpha \sum_{n=0}^\infty A_{n-m-1} \Delta^{\beta+1} \phi(n, t) \\
+ \sum_{p=\rho}^{\alpha} A_{n-p+1} \sum_{n=0}^\infty A_{n-r-1} \Delta \phi(n, t) \\
+ \sum_{n=0}^\infty A_{n-m} \Delta \phi(n, t) \\
= O(\rho^{\alpha}, \rho^{-\alpha}, t^{2\alpha-1}) + O(m^{\alpha}, m^{-\alpha}, t^{2\alpha-1}) + O(\rho^{\alpha-1}, \rho^{-\alpha}, t^{2\alpha-2}) \\
+ O(m^{\alpha-1}, m^{-\alpha}, t^{2\alpha-2}) + O(\rho^{-1}, t^{2\alpha-2}) \\
= O(t^{2\alpha-1}), 
\]

which is the required result.

**Lemma 5.** If the series \( \sum_{n=0}^\infty a_n \) with \( a_0 = 0 \) is evaluable \( |C, \alpha| \), \( \alpha > 0 \), to zero, then the series \( \sum_{n=0}^\infty a_n \phi(n, t) \) is convergent for any \( t > 0 \) and

\[ \sum_{n=1}^\infty a_n \phi(n, t) = \sum_{r=1}^\infty s_r \sum_{n=r}^\infty A_{n-r} \Delta \phi(n, t). \]

**Remark.** It is remarkable that in this lemma the series \( \sum_{n=0}^\infty a_n \cos nt \) need not be a Fourier series.

**Proof.** The summability \( |C, \alpha| \) of the series \( \sum_{n=0}^\infty a_n \) implies that the series \( \sum_{n=0}^\infty n^{-\alpha} |a_n| \) is convergent, by the well-known theorem. Hence, by (2.3), it follows that the series \( \sum_{n=0}^\infty a_n \phi(n, t) \) is convergent for any \( t > 0 \). By the repeated use of Abel’s transformation,

\[ \sum_{n=1}^m a_n \phi(n, t) = \sum_{n=1}^{m-1} s_n \Delta \phi(n, t) + \sum_{j=0}^{\beta} j s_{m-j} \phi(m - j, t), \]

where, by \( s_n^j = o(n^\alpha) \) when \( 0 \leq j \leq \alpha \) and (2.3),

\[ \sum_{j=0}^{\beta} j s_{m-j} \phi(m - j, t) = o(1) \quad \text{as} \quad m \to \infty. \]
Thus
\[ \sum_{n=1}^{\infty} a_n \phi(n, t) = \sum_{n=1}^{\infty} s_n^{\beta} \phi(n, t). \]

By the well-known formula
\[ s_n^{\beta} = \sum_{r=0}^{n} A_{n-r} \beta \phi(n, t) = \sum_{r=1}^{n} A_{n-r} \beta \phi(n, t), \]
we have formally
\[ \sum_{n=1}^{\infty} s_n^{\beta} \phi(n, t) = \sum_{n=1}^{\infty} \Delta^{\beta+1} \phi(n, t) \sum_{r=1}^{n} A_{n-r} \beta \phi(n, t) \]
\[ = \sum_{r=1}^{\infty} s_r^{-1} \sum_{n=r}^{\infty} A_{n-r} \beta \phi(n, t). \]

Thus, if we may prove that this interchange is legitimate, then the proof is complete. For this purpose, it is sufficient to prove that, for a fixed \( t > 0 \),
\[ \lim_{N \to \infty} \sum_{n=1}^{N} s_r^{-1} \sum_{n=N+1}^{\infty} A_{n-r} \beta \phi(n, t) = 0. \]

Since, by (2.3),
\[ \sum_{r=1}^{\infty} \Delta^{\beta+1} \phi(n, t) = \Delta^{\beta} \phi(n, t), \]
we have (2.8) for an integer \( \alpha \), using \( s_n^{\alpha} = o(n^{\alpha}) \) and (2.3). Hence we shall prove (2.8) in which \( \alpha \) is not an integer. By (2.2), (2.3) and (2.9), for \( \nu, 1 \leq \nu \leq N \),
\[ \sum_{n=N+1}^{\infty} A_{n-r} \beta \phi(n, t) \]
\[ = A_{N-r+1}^{\beta} \phi(N + 1, t) + \sum_{n=N+1}^{\infty} A_{n-r+1}^{\beta-1} \phi(n + 1, t) \]
\[ = O((N - \nu + 1) \beta^{n-\alpha} N^{-\alpha}). \]

Now the summability \( |c, \alpha| \) of the series \( \sum_{n=0}^{\infty} a_n \) implies
\[ \sum_{r=1}^{n} |s_r^{-1}| = O(n^\alpha), \]
by the series analogue of Rajagopal's Lemma [5, Lemma 10]. Then, let us write, putting \( \eta = [N/2] \),
\[
\sum_{r=1}^{N} \frac{a-1}{s_r} \sum_{n=N+1}^{\infty} A_{n-r}^{\alpha} \Delta^{\beta+1} \phi(n, t) = \left( \sum_{r=1}^{N} \sum_{n=r}^{N} \right) = I_1 + I_2,
\]
where, by (2.10) and (2.11),
\[
I_1 = O\left( \sum_{r=1}^{N} \frac{a-1}{s_r} \cdot (N - \nu + 1)^{\beta-\alpha} \cdot N^{-\alpha} \right)
= O\left( N^{\beta-2a} \sum_{r=1}^{N} \frac{a-1}{s_r} \right) = O(N^{\beta-\alpha}) = o(1).
\]

For \( I_2 \), we take an arbitrarily fixed number \( L \), but large enough. Then, using (2.10), (2.11) and \( s_n^{-1} = o(n) \),
\[
I_2 = O\left( N^{-\alpha} \sum_{r=1}^{N} \frac{a-1}{s_r} \cdot (N - \nu + 1)^{\beta-\alpha} \right)
= O\left( N^{-\alpha} \sum_{r=1}^{L} \frac{a-1}{s_{N-r+1}} \cdot N^{-\alpha} \right) + O\left( N^{-\alpha} \sum_{r=L}^{\infty} \frac{a-1}{s_{N-r+1}} \cdot L^{-\alpha} \right)
= o\left( \sum_{r=1}^{L} \nu^{\beta-\alpha} \right) + O\left( N^{-\alpha} \sum_{r=1}^{L} \frac{a-1}{s_{N-r+1}} \cdot L^{-\alpha} \right)
+ O\left( N^{-\alpha} \sum_{r=1}^{N-\nu} \frac{a-1}{s_{N-r+1}} \cdot (N - \eta + 1)^{\beta-\alpha} \right)
+ O\left( N^{-\alpha} \sum_{r=L}^{\infty} \left( \sum_{n=n+1}^{\infty} \frac{a-1}{s_{N-n+1}} \right) \cdot \nu^{\beta-\alpha-1} \right)
= o(1) + O(L^{\beta-\alpha}) + O(N^{\beta-\alpha}) + O(L^{\beta-\alpha})
= o(1) + O(L^{\beta-\alpha}).
\]

Since \( L \) is arbitrary, we have \( I_2 = o(1) \) by \( \beta - \alpha < 0 \). Thus (2.8) in which \( \alpha \) is not an integer is valid.

3. Proof of theorem. The notations in the former paragraphs are used without further explanations. We shall now prove the theorem in which \( \alpha \) is not an integer, the remaining part\(^1\) being obtained by the analogous method. In this case, by the Chandrasekharan and Szász Theorem [3, Theorem 5] for the proof, it is sufficient to prove that

\(^1\) This part is obtained as a corollary of Obreschkoff's Theorem [4, Satz 2] using the following Chandrasekharan and Szász Theorem. This remark is due to Professor G. Sunouchi.
\[ \Phi_n(t) = \int_0^t (t^2 - u^2)^{a-1}f(u)du = o(t^{2a-1}). \]

Since \( \alpha \geq 1 \), we have

\[ \Phi_n(t) = \sum_{n=1}^{\infty} a_n \phi(n, t). \]

By Lemma 5 and the Abel transformation, using (2.5), (2.6) and \( s_n^* = o(n^\alpha) \),

\[ \sum_{n=1}^{\infty} a_n \phi(n, t) = \sum_{r=1}^{\infty} \sigma_r \sum_{n=1}^{\infty} A_{n-r}^{a} \phi(n, t) \]

\[ = \sum_{r=1}^{\infty} \sigma_r \left( A_r^{a} \sum_{n=1}^{\infty} A_{n-r}^{a} \phi(n, t) \right) \]

\[ = \sum_{r=1}^{\infty} (\sigma_r - \sigma_{r+1}) U_r(t), \]

where \( \sigma_r^* = s_r^*/A_r^a \). Since the series is evaluable \( |C, \alpha| \), that is,

\[ \sum_{r=1}^{\infty} |\sigma_r - \sigma_{r+1}| < + \infty, \]

using (2.6), for an arbitrary positive number \( \epsilon \), there exists an \( N = N(\epsilon) \) such that

\[ \left| \sum_{r=1}^{N} (\sigma_r - \sigma_{r+1}) U_r(t) \right| = O \left( t^{2a-1} \sum_{r=1}^{\infty} |\sigma_r - \sigma_{r+1}| \right) < \epsilon t^{2a-1}. \]

On the other hand, for a fixed \( \nu > 0 \), by (2.3),

\[ \sum_{n=1}^{\infty} A_{n-r}^{a} \phi(n, t) = O \left( t^{\sigma+\beta+1} \sum_{n=1}^{\infty} (n - \nu + 1)^{\beta-a} \right) \]

\[ = O(t^{\sigma+\beta+1}). \]

Then, for \( m \leq N \),

\[ U_m(t) = \sum_{r=1}^{m} A_r^{a} \sum_{n=1}^{m} A_{n-r}^{a} \phi(n, t) = O \left( t^{\sigma+\beta+1} \sum_{r=1}^{m} A_r^{a} \right) = O(t^{\sigma+\beta+1}N^{\sigma+1}). \]

Hence

\[ \lim_{t \to 0+} \sum_{r=1}^{N} (\sigma_r - \sigma_{r+1}) U_r(t) t^{2a-1} = 0. \]
Therefore we have

\[ \limsup_{t \to 0^+} \left| \sum_{r=1}^{\infty} (\sigma_r^\alpha - \sigma_{r+1}^\alpha) U_r(t)/t^{2\alpha-1} \right| \leq \varepsilon. \]

Since \( \varepsilon \) is arbitrary, we get

\[ \lim_{t \to 0^+} \sum_{r=1}^{\infty} (\sigma_r^\alpha - \sigma_{r+1}^\alpha) U_r(t)/t^{2\alpha-1} = 0, \]

that is,

\[ \lim_{t \to 0^+} \Phi_\alpha^*(t)/t^{2\alpha-1} = 0, \]

and the theorem in which \( \alpha \) is not an integer is completely proved.

4. **Bessel summability.** A series \( \sum_{n=1}^{\infty} c_n \) is said to be evaluable \((J_\nu)\) to zero if the series

\[ \sum_{n=1}^{\infty} c_n \psi(\mu, nt) \]

converges in some interval \( 0 < t < t_0 \), and if its sum tends to zero as \( t \to 0 \). From the argument in §3, we have, by (2.4),

**Corollary.** A series evaluable \( |C, \alpha| \) to zero is also evaluable \((J_{\alpha-1/2})\) to zero when \( \alpha \geq 1 \).

**References**


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