A NOTE ON HOMOGENEOUS COMPLEX CONTACT MANIFOLDS

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1. Introduction. A complex contact manifold is by definition a complex analytic manifold of odd dimension \( 2n+1 \) over which is defined to within an analytic scalar multiple an analytic pfaffian form \( \omega \) of class \( 2n+1 \), i.e. such that \( \omega \wedge d\omega^* \neq 0 \) at any point. Such a manifold is given \([4]\) by a covering by coordinate neighborhoods \( \{ U_i \} \) on each of which is defined a form \( \omega_i \) of the type described with \( \omega_i = f_i \omega_j \) whenever \( U_i \cap U_j \neq \emptyset \), \( f_{ij} \) being nonvanishing, analytic scalar functions. A natural question arises as to whether the contact form may be chosen globally, that is the \( \omega_i \) so chosen that each \( f_{ij} = 1 \). It was shown by S. Kobayashi \([4]\) that this can be done if and only if the first Chern class, \( c_1(M) \), vanishes. Here and in what follows we will suppose \( M \) to be compact. Under the additional assumption that \( M \) is simply connected and homogeneous both with respect to the complex and the contact structure the author \([2]\) characterized and enumerated those manifolds for which \( c_1(M) = 0 \). In what follows it is shown that these additional assumptions already imply \( c_1(M) = 0 \), so that combining this with the results of \([2]\) we obtain:

Theorem I. Let \( M \) be a compact, simply connected, homogeneous complex contact manifold containing more than one point. Then \( c_1(M) \neq 0 \), hence \( \omega \) can not be globally defined; \( M \) is Kählerian; and there is exactly one such manifold corresponding to each of the classes of simple Lie groups \( A_n, B_n, C_n, \) and \( D_n \) and the five exceptional simple groups. No other manifolds satisfying these hypotheses exist.

When we make no homogeneity assumption, the classical example of such a manifold is the bundle of complex "co-directions" over a complex analytic manifold of dimension \( n+1 \): the fibres are complex projective spaces of dimension \( n \), and with local coordinates \( z^1, \ldots, z^{n+1} \) in the base space, \( p_1, \ldots, p_{n+1} \) (homogeneous) coordinates in the fibre we have \( \omega = p_1dz^1 + \cdots + p_{n+1}dz^{n+1} \). In this connection the question arises as to whether the spaces of Theorem I are already included in this example. This question is answered by the following:

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2 Throughout dimension refers to complex dimension and analytic means complex analytic.
Theorem II. With the exception of the manifolds corresponding to \( A_n \), none of the manifolds of Theorem I can be homeomorphic to a bundle of complex "co-directions" over a complex manifold, i.e. to an example of the type described above.

2. Proof of Theorem I. Following H. C. Wang [5], we will refer to a compact, simply connected, homogeneous complex manifold as a \( C \)-space. The notation used in this section is that of [5] and [2] and for brevity will not be redefined. Suppose \( G/L \) to be a \( C \)-space with a globally defined contact form \( \omega \) which is \( G \)-invariant. Without loss of generality we suppose \( G \) to be complex semi-simple. By proceeding as in the real case [3, §4] we will arrive at a contradiction which shows the impossibility of a globally defined contact form. This together with the results of [2] gives Theorem I. Let 

\[ K = \{ g \in G \mid \text{ad}(g)^* \omega^* = \omega^* \} \]

where \( \text{ad}(g)^* \) denotes the transformation on forms on the Lie algebra \( \hat{G} \) induced by the inner automorphism \( \text{ad}(g) : G \to G \) and \( \omega^* = p^* \omega \), \( p : G \to G/L \) being the natural map. Clearly \( K \) is a closed subgroup and contains \( L \). As in the real case we see \( \hat{K} = \{ X \in \hat{G} \mid d\omega^*(X, \hat{G}) = 0 \} \) and therefore \( \hat{K} \) is a complex Lie algebra; hence \( K \) is a closed, complex subgroup and \( G/K \) a homogeneous complex manifold. There is a natural analytic map of \( G/L \) onto \( G/K \) so we know \( G/K \) to be compact and connected. If \( K_0 \) is the identity component of \( K \), then the same statements apply to \( K_0 \) and \( G/K_0 \) provided \( L \) is connected, which is, in fact, the case since \( G \) is connected and \( G/L \) is simply connected. Then from the homotopy sequence of \( G/L \to G/K_0 \) it follows that \( G/K_0 \) is simply connected and hence is a \( C \)-space. It contains more than a single point for, as in the real case, \( \omega \wedge d\omega^* \neq 0 \) implies \( \dim \hat{K} = 1 + \dim \hat{L} \) and, if \( G/K_0 \) is a single point, then \( K_0 = G \) and \( K_0/L \) is an abelian group of complex dimension one. Since it is compact and simply connected, it too contains but one point which is contrary to assumption.

Now let \( Z \in \hat{G} \) correspond to \( \omega^* \) relative to the Killing form \( (X, Y) \) on \( \hat{G} \). Then \( \omega^*(X) = (Z, X) \) and \( d\omega^*(X, Y) = \omega([X, Y])/2 \). Thus \( d\omega^*(X, Y) = 0 \) if and only if \( ([Z, X], Y) = (Z, [X, Y]) = 0 \). Since the Killing form is nondegenerate, the first of these vanishes for all \( Y \in \hat{G} \) if and only if \( [X, Z] = 0 \), therefore \( \hat{K} = \hat{C}(Z) \), the centralizer of \( Z \), i.e. \( \hat{C}(Z) = \{ X \mid [X, Z] = 0 \} \). The desired contradiction is then obtained from the following:

Lemma. Let \( G \) be a connected, complex, semi-simple Lie group and \( K \) a closed, connected, complex subgroup whose Lie algebra \( \hat{K} \) is the centralizer of some \( Z \in \hat{G} \). Then, if \( G/K \) is a \( C \)-space, it contains only one point.

Proof. According to Wang [5], if \( G/K \) is a \( C \)-space, then for the
Lie algebras we have the following situation. There exists a Cartan subalgebra $\hat{H} \subset \hat{G}$ with rational basis $h_1, \ldots, h_b, h_{b+1}, \ldots, h_a, h_{a+1}, \ldots, h_1$, with $b = 2u$, where $\hat{K}$ has as basis $k_1, \ldots, k_u, h_{b+1}, \ldots, h_a, h_{a+1}, \ldots, h_1; E_{\pm a_1}, \ldots, E_{\pm a_r}, E_{e_1}, \ldots, E_{e_r}$, where $k_1 = i h_1 + c h_2$, $k_2 = ih_2 + ch_4$, $\ldots$, $k_u = ih_{b-1} + ch_b$, $c$ a real number, and $\alpha_1, \ldots, \alpha_r, \sigma_1, \ldots, \sigma_r$ are all positive roots and $\pm \alpha_i, \ldots, \pm \alpha_r$ are all those roots which vanish on each of the vectors $h_1, \ldots, h_a$. The maximal semi-simple subalgebra $\hat{Q}$ of $\hat{K}$ is spanned by $h_{a+1}, \ldots, h_i$ and $E_{\pm a_1}, \ldots, E_{\pm a_r}$. Since $Z \in \hat{K}$, we may write with suitable $h \in \hat{H} \cap \hat{K}$, $Z = h + \sum_{i=1}^{a} a_i E_{a_i} + \sum_{j=1}^{b} b_j E_{-a_j} + \sum_{f=1}^{r} d_j E_{e_f}$. We shall prove $Z = 0$, whence $\hat{G} = \hat{K}$ and $G/K$ is a single point. If $X \in \hat{K}$ is either in $\hat{H}$ or a root vector $E_i$, then since $[A, Z] = 0$, each summand of $[X, Z] + \sum_{i=1}^{a} a_i [X, E_{a_i}] + \sum_{j=1}^{b} b_j [X, E_{-a_j}] + \sum_{f=1}^{r} d_j [X, E_{e_f}]$ vanishes since each lies in a different summand of a direct sum decomposition of $\hat{G}$. Thus $[E_i, h] = \sigma(h) E_i$ vanishes for each positive root $\sigma$, hence $h = 0$. Multiplying by $E_{+a}$ or $E_{-a}$ we obtain $a_i = 0 = b_i$ for each $i$. Finally let $\sigma_j$ be one of the roots $\sigma_1, \ldots, \sigma_r$ and $k, 1 \leq k \leq a$, be such that $\sigma_j(h_k) \neq 0$. There must be at least one such or $\sigma_j$ would be among the $\pm \alpha_i$. If $b+1 \leq k \leq a$, then, since $h_k \in \hat{K}$ and thus $[h_k, Z] = 0$, we at once get $d_{j_0}(h_k, E_{e_{j_0}}) = \sigma_{j_0}(h_k)d_{j_0}E_{e_{j_0}} = 0$ and thus $d_{j_0} = 0$. But, if $1 \leq k \leq b$, then either $[ih_k + ch_{k+1}, Z] d_{j_0} = 0$ or $[ih_{k-1} + ch_k, Z] d_{j_0} = 0$ according to the parity of $k$. In either case $d_{j_0} = 0$ unless the term in brackets vanishes. But this cannot happen since, say in the first case, vanishing of the term in brackets would imply $\sigma_{j_0}(h_k) + c \sigma_{j_0}(h_{k+1}) = 0$ and thus since $\sigma_{j_0}$ is real on $h_k$ and $h_{k+1}$, $\sigma_{j_0}(h_k) = 0 = \sigma_{j_0}(h_{k+1})$, contrary to our choice of $h_k$. A corresponding argument applies to the other case and to each of the $\sigma_j$, $j = 1, \ldots, r$. Thus each $d_j = 0$ and so $Z = 0$ as was claimed.

3. Proof of Theorem II. We now consider the possibility that the homogeneous complex contact manifold $M = G/L$ might be a bundle of complex co-directions over a complex analytic manifold $B$ of dimension $n + 1$. In this case the fibre $F$ would be a complex projective space of dimension $n$. By Theorem I $M$ is a Kähler manifold, $F$ is also Kählerian, and by Blanchard [1, Prop. II.2, p. 184] we see that $B$ is Kählerian. Concerning the topology of $M$, $F$, and $B$ we use the fact that $H^1(F) = 0$, which implies that the transgression $H^1(F) \to H^2(B)$ vanishes, together with the fact that $\pi_1(B) = 0$, which follows from the fact that $F$ and $M$ are both simply connected spaces and, of course, implies that $\pi_1(B)$ acts trivially on the cohomology groups of $F$. This allows us to use Blanchard, op. cit., Theorem II 1.2, p. 178, according to which the real cohomology of $M$ is isomorphic...
to that of $B \times F$. In particular for the Poincaré polynomials of $M$, $F$, and $B$ we have

$$P_M(t) = P_F(t)P_B(t) = (1 + t^2 + \cdots + t^n)(1 + at^2 + \cdots + t^{n+1})$$

where $a \geq 1$ since $B$ is Kählerian. It follows that the second Betti number of $M$, $p_2(M) \geq 2$. Now let $M = G/L$, $G$ complex semi-simple as before. It is known that $\pi_3(G) = 0$. The homotopy sequence of the bundle $G \to G/L$ yields the exact sequence:

$$0 \to \pi_2(G/L) \to \pi_1(L) \to \pi_1(G) \to 0,$$

which, in particular, says that the rank of $\pi_3(G/L)$ is no greater than the rank of $\pi_1(L)$. Since $G/L$ is simply connected, $\pi_2(G/L) \cong H^2(G/L)$; thus $p_2(M) \leq \text{rank } \pi_1(L)$. Now examining the individual cases in Theorem I and replacing $G$ and $L$ by their maximal compact subgroups $G'$, $L'$, cf. [2], shows that $L' = S U(n)$ where $S U(n)$ is semi-simple and thus rank $\pi_1(L) = r$. Since $2 \leq p_2(M) \leq r$ we must have $r \geq 2$, but this occurs only in the case corresponding to $A_n$ where $G' = S U(n + 1)$ and $L' = S U(n - 1) \times T^2$.

4. Remarks. I. Since every homogeneous complex contact manifold, at least if it is simply connected, is a Kähler manifold, one might ask if this is true in general. However, from Blanchard, op. cit., we see that if $B$ is a non-Kählerian, compact, complex analytic manifold then no bundle over it can be Kählerian. Thus the bundle of co-directions furnishes an example of a non-Kählerian compact complex contact manifold. It will be simply connected if $B$ is chosen simply connected.

II. In the one exceptional case of Theorem II we have $S U(n + 1) \supset S U(n) \times T^1 \supset S U(n - 1) \times T^2$ so that denoting $S U(n + 1)/S U(n - 1)$ by $M$, $S U(n + 1)/S U(n) \times T^1$ by $B$ and $S U(n) \times T^1/S U(n)$ by $F$ we see that $M$ is a bundle over $B$ whose fibre $F$ is a projective space $P_{n-1}(C)$ of dimension $n - 1$ and whose base $B$ is a projective space $P_n(C)$ of dimension $n$. And in fact the bundle is homeomorphic to the bundle of tangent co-directions to $n$ dimensional complex projective space.

A proof of this last statement may be sketched as follows. We regard $P_n(C)$ as the space of directions through the origin in $C_{n+1}$, $n + 1$ dimensional Hermitian space, and the complex $n$-plane orthogonal to a given direction as the tangent space to the corresponding point of $P_n(C)$. Then $S U(n + 1)$ is transitive on the pairs $(x, d) = (\text{point } x, \text{tangent direction at } x)$ of $P_n(C)$. The subgroups fixing respectively $x$ and $(x, d)$ are $S U(n) \times T^1$ and $S U(n - 1) \times T^2$. Then
the result follows easily if we note that the space of tangent co-directions to $P_n(C)$ is homeomorphic to the space of tangent directions as we may see by choosing a Hermitian metric on $P_n(C)$.

BIBLIOGRAPHY


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