A NOTE ON HOMOGENEOUS COMPLEX CONTACT MANIFOLDS

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1. Introduction. A complex contact manifold is by definition a complex analytic manifold of odd dimension $2n+1$ over which is defined to within an analytic scalar multiple an analytic pfaffian form $\omega$ of class $2n+1$, i.e. such that $\omega \wedge d\omega^n \neq 0$ at any point. Such a manifold is given [4] by a covering by coordinate neighborhoods $\{ U_i \}$ on each of which is defined a form $\omega_i$ of the type described with $\omega_i = f_i \omega_j$ whenever $U_i \cap U_j \neq \emptyset$, $f_{ij}$ being nonvanishing, analytic scalar functions. A natural question arises as to whether the contact form may be chosen globally, that is the $\omega_i$ so chosen that each $f_{ij} = 1$. It was shown by S. Kobayashi [4] that this can be done if and only if the first Chern class, $c_1(M)$, vanishes. Here and in what follows we will suppose $M$ to be compact. Under the additional assumption that $M$ is simply connected and homogeneous both with respect to the complex and the contact structure the author [2] characterized and enumerated those manifolds for which $c_1(M) \neq 0$. In what follows it is shown that these additional assumptions already imply $c_1(M) \neq 0$, so that combining this with the results of [2] we obtain:

**Theorem I.** Let $M$ be a compact, simply connected, homogeneous complex contact manifold containing more than one point. Then $c_1(M) \neq 0$, hence $\omega$ can not be globally defined; $M$ is Kählerian; and there is exactly one such manifold corresponding to each of the classes of simple Lie groups $A_n$, $B_n$, $C_n$, and $D_n$ and the five exceptional simple groups. No other manifolds satisfying these hypotheses exist.

When we make no homogeneity assumption, the classical example of such a manifold is the bundle of complex “co-directions” over a complex analytic manifold of dimension $n+1$: the fibres are complex projective spaces of dimension $n$, and with local coordinates $z^1, \ldots, z^{n+1}$ in the base space, $p_1, \ldots, p_{n+1}$ (homogeneous) coordinates in the fibre we have $\omega = p_1 dz^1 + \cdots + p_{n+1} dz^{n+1}$. In this connection the question arises as to whether the spaces of Theorem I are already included in this example. This question is answered by the following:

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* Throughout dimension refers to complex dimension and analytic means complex analytic.
Theorem II. With the exception of the manifolds corresponding to $A_n$, none of the manifolds of Theorem I can be homeomorphic to a bundle of complex "co-directions" over a complex manifold, i.e. to an example of the type described above.

2. Proof of Theorem I. Following H. C. Wang [5], we will refer to a compact, simply connected, homogeneous complex manifold as a C-space. The notation used in this section is that of [5] and [2] and for brevity will not be redefined. Suppose $G/L$ to be a C-space with a globally defined contact form $\omega$ which is $G$-invariant. Without loss of generality we suppose $G$ to be complex semi-simple. By proceeding as in the real case [3, §4] we will arrive at a contradiction which shows the impossibility of a globally defined contact form. This together with the results of [2] gives Theorem I. Let $K = \{ g \in G | \text{ad}(g)^* \omega = \omega^* \}$ where $\text{ad}(g)^*$ denotes the transformation on forms on the Lie algebra $\mathfrak{g}$ induced by the inner automorphism $\text{ad}(g) : G \to G$ and $\omega^* = \rho^* \omega$, $\rho : G \to G/L$ being the natural map. Clearly $K$ is a closed subgroup and contains $L$. As in the real case we see $\hat{K} = \{ X \in \mathfrak{g} | d\omega^*(X, \mathfrak{g}) = 0 \}$ and therefore $\hat{K}$ is a complex Lie algebra; hence $K$ is a closed, complex subgroup and $G/K$ a homogeneous complex manifold. There is a natural analytic map of $G/L$ onto $G/K$ so we know $G/K$ to be compact and connected. If $K_0$ is the identity component of $K$, then the same statements apply to $K_0$ and $G/K_0$ provided $L$ is connected, which is, in fact, the case since $G$ is connected and $G/L$ is simply connected. Then from the homotopy sequence of $G/L \to G/K_0$ it follows that $G/K_0$ is simply connected and hence is a C-space. It contains more than a single point for, as in the real case, $\omega \wedge d\omega^* \neq 0$ implies $\dim \hat{K} = 1 + \dim \hat{L}$ and, if $G/K_0$ is a single point, then $K_0 = G$ and $K_0/L$ is an abelian group of complex dimension one. Since it is compact and simply connected, it too contains but one point which is contrary to assumption.

Now let $Z \in \hat{G}$ correspond to $\omega^*$ relative to the Killing form $(X, Y)$ on $\hat{G}$. Then $\omega^*(X) = (Z, X)$ and $d\omega^*(X, Y) = \omega([X, Y])/2$. Thus $d\omega^*(X, Y) = 0$ if and only if $([Z, X], Y) = (Z, [X, Y]) = 0$. Since the Killing form is nondegenerate, the first of these vanishes for all $Y \in \hat{G}$ if and only if $[X, Z] = 0$; therefore $\hat{K} = \hat{C}(Z)$, the centralizer of $Z$, i.e. $\hat{C}(Z) = \{ X | [X, Z] = 0 \}$. The desired contradiction is then obtained from the following:

Lemma. Let $G$ be a connected, complex, semi-simple Lie group and $K$ a closed, connected, complex subgroup whose Lie algebra $\hat{K}$ is the centralizer of some $Z \in \hat{G}$. Then, if $G/K$ is a C-space, it contains only one point.

Proof. According to Wang [5], if $G/K$ is a C-space, then for the
Lie algebras we have the following situation. There exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with rational basis $h_1, \ldots, h_b, h_{b+1}, \ldots, h_a, h_{a+1}, \ldots, h_1$, with $b = 2u$, where $\mathfrak{h}$ has as basis $k_1, \ldots, k_u, h_{b+1}, \ldots, h_a, h_{a+1}, \ldots, h_1; E_{2a_1}, \ldots, E_{2a_t}, E_{e_1}, \ldots, E_{e_r}$, where $k_i = ih_1 + c h_2$, $k_2 = ih_3 + c h_4, \ldots, k_u = ih_{b-1} + c h_b, c$ a real number, and $\alpha_1, \ldots, \alpha_s$, $\sigma_1, \ldots, \sigma_r$ are all positive roots and $\pm \alpha_1, \ldots, \pm \alpha_s$ are all those roots which vanish on each of the vectors $h_1, \ldots, h_a$. The maximal semi-simple subalgebra $\mathfrak{g}$ of $\mathfrak{h}$ is spanned by $h_{a+1}, \ldots, h_1$ and $E_{2a_1}, \ldots, E_{2a_t}$.

Since $Z \in \mathfrak{h}$, we may write with suitable $h \in \mathfrak{h} \cap \mathfrak{g}$, $Z = h + \sum_{j=1}^{i} a_i E_{a_i} + \sum_{i=t}^{r} b_i E_{-a_i} + \sum_{j=1}^{s} d_j E_{e_j}$. We shall prove $Z = 0$, whence $\mathfrak{g} = \mathfrak{h}$ and $\mathfrak{g}/\mathfrak{k}$ is a single point. If $X \in \mathfrak{h}$ is either in $\mathfrak{h}$ or a root vector $E_\sigma$, then since $[X, Z] = 0$, each summand of $[X, h] + \sum_{i=1}^{t} a_i [X, E_{a_i}] + \sum_{i=t-1}^{r} b_i [X, E_{-a_i}] + \sum_{j=1}^{s} d_j [X, E_{e_j}]$ vanishes since each lies in a different summand of a direct sum decomposition of $\mathfrak{h}$. Thus $[E_\sigma, h] = \sigma(h) E_\sigma$ vanishes for each positive root $\sigma$, hence $h = 0$. Multiplying by $E_{a_1}$ or $E_{-a}$ we obtain $a_i = 0 = b_i$ for each $i$. Finally let $\sigma_{j_0}$ be one of the roots $\sigma_1, \ldots, \sigma_r$ and $1 \leq k \leq a$, be such that $\sigma_{j_0}(h_k) \neq 0$. There must be at least one such or $\sigma_{j_0}$ would be among the $\pm \alpha_i$. If $b + 1 \leq k \leq a$, then, since $h_k \in \mathfrak{h}$ and thus $[h_k, Z] = 0$, we at once get $d_{j_0}(h_k, E_{e_{j_0}}) = \sigma_{j_0}(h_k) d_{j_0}(h_k) E_{e_{j_0}} = 0$ and thus $d_{j_0} = 0$. But, if $1 \leq k \leq b$, then either $[ih_k + c h_{k+1}, Z] d_{j_0} = 0$ or $[ih_{k-1} + c h_k, Z] d_{j_0} = 0$ according to the parity of $k$. In either case $d_{j_0} = 0$ unless the term in brackets vanishes. But this cannot happen since, say in the first case, vanishing of the term in brackets would imply $\sigma_{j_0}(h_k) + c \sigma_{j_0}(h_{k+1}) = 0$ and thus since $\sigma_{j_0}$ is real on $h_k$ and $h_{k+1}$, $\sigma_{j_0}(h_k) = 0 = \sigma_{j_0}(h_{k+1})$, contrary to our choice of $h_k$. A corresponding argument applies to the other case and to each of the $\sigma_j, j = 1, \ldots, r$. Thus each $d_{j_0} = 0$ and so $Z = 0$ as was claimed.

3. Proof of Theorem II. We now consider the possibility that the homogeneous complex contact manifold $M = G/L$ might be a bundle of complex co-directions over a complex analytic manifold $B$ of dimension $n + 1$. In this case the fibre $F$ would be a complex projective space of dimension $n$. By Theorem I $M$ is a Kähler manifold, $F$ is also Kählerian, and by Blanchard [1, Prop. II.2, p. 184] we see that $B$ is Kählerian. Concerning the topology of $M$, $F$, and $B$ we use the fact that $H^1(F) = 0$, which implies that the transgression $H^1(F) \rightarrow H^2(B)$ vanishes, together with the fact that $\pi_1(B) = 0$, which follows from the fact that $F$ and $M$ are both simply connected spaces and, of course, implies that $\pi_1(B)$ acts trivially on the cohomology groups of $F$. This allows us to use Blanchard, op. cit., Theorem II 1.2, p. 178, according to which the real cohomology of $M$ is isomorphic
to that of \( B \times F \). In particular for the Poincaré polynomials of \( M, F, \) and \( B \) we have
\[
P_M(t) = P_F(t)P_B(t) = (1 + t^2 + \cdots + t^a)(1 + at^2 + \cdots + t^{a+1})
\]
where \( a \geq 1 \) since \( B \) is Kählerian. It follows that the second Betti number of \( M, p_2(M) \geq 2 \). Now let \( M = G/L, G \) complex semi-simple as before. It is known that \( \pi_2(G) = 0 \). The homotopy sequence of the bundle \( G \to G/L \) yields the exact sequence:
\[
0 \to \pi_2(G/L) \to \pi_1(L) \to \pi_1(G) \to 0,
\]
which, in particular, says that the rank of \( \pi_2(G/L) \) is no greater than the rank of \( \pi_1(L) \). Since \( G/L \) is simply connected, \( \pi_2(G/L) \approx H^2(G/L) \); thus \( p_2(M) \leq \text{rank } \pi_1(L) \). Now examining the individual cases in Theorem I and replacing \( G \) and \( L \) by their maximal compact subgroups \( G', L' \), cf. [2], shows that \( L' = \mathbb{S}^1 \times F \), where \( F \) is semi-simple and thus \( \text{rank } \pi_1(L) = r \). Since \( 2 \leq p_2(M) \leq r \) we must have \( r = 2 \), but this occurs only in the case corresponding to \( G' = SU(n+1) \) and \( L' = SU(n-1) \times T^2 \).

4. Remarks. I. Since every homogeneous complex contact manifold, at least if it is simply connected, is a Kähler manifold, one might ask if this is true in general. However, from Blanchard, op. cit., we see that if \( B \) is a non-Kählerian, compact, complex analytic manifold then no bundle over it can be Kählerian. Thus the bundle of co-directions furnishes an example of a non-Kählerian compact complex contact manifold. It will be simply connected if \( B \) is chosen simply connected.

II. In the one exceptional case of Theorem II we have \( SU(n+1) \supset SU(n) \times T \supset SU(n-1) \times T^2 \) so that denoting \( SU(n+1)/SU(n-1) \times T^2 \) by \( M, SU(n+1)/SU(n) \times T^1 \) by \( B \) and \( SU(n) \times T^1/SU(n-1) \times T^2 \) by \( F \) we see that \( M \) is a bundle over \( B \) whose fibre \( F \) is a projective space \( P_{n-1}(C) \) of dimension \( n-1 \) and whose base is a projective space \( P_n(C) \) of dimension \( n \). And in fact the bundle is homeomorphic to the bundle of tangent co-directions to \( n \) dimensional complex projective space.

A proof of this last statement may be sketched as follows. We regard \( P_n(C) \) as the space of directions through the origin in \( \mathbb{C}^{n+1}, n+1 \) dimensional Hermitian space, and the complex \( n \)-plane orthogonal to a given direction as the tangent space to the corresponding point of \( P_n(C) \). Then \( SU(n+1) \) is transitive on the pairs \( (x, d) = (\text{point } x, \text{tangent direction at } x) \) of \( P_n(C) \). The subgroups fixing respectively \( x \) and \( (x, d) \) are \( SU(n) \times T^1 \) and \( SU(n-1) \times T^2 \). Then
the result follows easily if we note that the space of tangent co-directions to $P_n(C)$ is homeomorphic to the space of tangent directions as we may see by choosing a Hermitian metric on $P_n(C)$.

**Bibliography**


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