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A MULTI-POINT GENERALIZATION OF NEWTON'S DIVIDED DIFFERENCE FORMULA

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Newton's divided difference formula for the polynomial of degree $n-1$ determined by the values of the function $f(x)$ and its divided differences up to the $(n-1)$th order originating from any single point, has the following multi-point generalization:

THEOREM I. *The unique interpolation polynomial $P(x)$, of degree $\sum_{i=1}^{n} r_i - 1$, which is determined by $f(x_i) \equiv f_i \equiv \mathfrak{D}^0 f_i$ and the first $r_i - 1$ divided differences, $\mathfrak{D}^m f_i \equiv [x_i x_{i_1} \cdots x_{i_m}]$, $m = 1, \cdots, r_i - 1$, originating from the n points x_i, $i = 1, \cdots, n$, is expressible as*

$$(1) \qquad P(x) = \sum_{i=1}^{n} \Pi_i(x)\psi_i(x), \text{ where}$$

$$(2) \qquad \Pi_i(x) = \prod_{j=1, j\neq i}^{n} (x - x_j)(x - x_{j_1}) \cdots (x - x_{j_{r_j - 1}}),$$

$$(3) \qquad \psi_i(x) = \sum_{m=0}^{r_i - 1} [x_i x_{i_1} \cdots x_{i_m}]\psi_{i,m}(x), \text{ and}$$

$$(4) \qquad \psi_{i,m}(x) = \sum_{s=0}^{r_i - m - 1} [x_{i_m} x_{i_{m+1}} \cdots x_{i_{m+s}}]_{1/\Pi_i}(x - x_i)(x - x_{i_1}) \cdots (x - x_{i_{m+s-1}}).$$

In (3) *and* (4), $x_{i_0} \equiv x_i$; *in* (3), $[x_i] \equiv f_i$; *in* (4), $[x_{i_m}]_{1/\Pi_i} \equiv 1/\Pi_i(x_{i_m})$, *the sth order divided difference* $[x_{i_m} x_{i_{m+1}} \cdots x_{i_{m+s}}]_{1/\Pi_i}$ *is for the function* $1/\Pi_i(x)$, *and* $\psi_{i,0}(x)$ *begins* $[x_i]_{1/\Pi_i} + \cdots$.

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Theorem I is a generalization of the author's previous result involving advancing differences of $f(x)$, at equal intervals, at each x_i [1]. Also, Theorem I extends the result in [1] beyond generalization to irregular distribution of points $(x_i, x_{i_1}, \cdots, x_{i_{r_i-1}})$ in that the coefficients of $(x-x_i)(x-x_{i_1})\cdots(x-x_{i_{m+s-1}})$ in (4) are given explicitly as $[x_{i_m}x_{i_{m+1}}\cdots x_{i_{m+s}}]_{1/\Pi_i}$, whereas in [1] corresponding quantities that are proportional to those coefficients are given implicitly as solutions of a triangular linear system.

Theorem I, which may be regarded as a partitioned form of Newton's divided difference formula, might be applied advantageously, say in reducing the computing error by employing divided differences of lower orders, or where the points $(x_i, x_{i_1}, \cdots, x_{i_{r_i-1}})$ are at the same irregular intervals, so that the divided differences $[x_{i_m}x_{i_{m+1}}\cdots x_{i_{m+s}}]_{1/\Pi_i}$ are the same linear combinations of $1/\Pi_i(x_{i_{m+t}})$, $t=0, 1, \cdots, s$, for every i.

Proof of Theorem I. We use the following formula [2] for the mth divided difference of any $F(x)$ for the $m+1$ points $x_0, x_1, \cdots, x_m$:

$$\mathfrak{D}^m F \equiv [x_0x_1\cdots x_m]_F = \sum_{i=0}^{m}\left\{F(x_i)\Big/\prod_{j=0, j\neq i}^{m}(x_i-x_j)\right\}. \tag{5}$$

To show $\mathfrak{D}^m P(x)\big|_{x=x_i} = [x_ix_{i_1}\cdots x_{i_m}]$, $m=0, 1, \cdots, r_i-1$, we note first from (5) and (2) that the divided difference operator $\mathfrak{D}^m\cdots\big|_{x=x_i}$ annihilates every term in (1) except the ith. It suffices to show that

$$\mathfrak{D}^{m'}\Pi_i(x)\psi_{i,m}(x)\big|_{x=x_i} = \delta_m^{m'}, \qquad m' = 0, 1, \cdots, r_i-1. \tag{6}$$

For $m' \leqq m$, the result follows from (4) and (5), for arbitrary values of the coefficients of $(x-x_i)(x-x_{i_1})\cdots(x-x_{i_{m+s-1}})$ in (4) beyond the first.

We prove (6) for any $m'>m$ by induction, assuming (4) as far as $[x_{i_m}x_{i_{m+1}}\cdots x_{i_{m+s-1}}]_{1/\Pi_i}(x-x_i)(x-x_{i_1})\cdots(x-x_{i_{m+s-2}})$ to be equivalent to (6) for $m' \leqq m+s-1$, which was just shown for $s=1$. For (6) to hold when $m' = m+s$, denoting in (4) the coefficient of $(x-x_i)(x-x_{i_1})\cdots(x-x_{i_{m+s-1}})$ by $a_{i_m}^{(s)}$, application of (5) to $[x_ix_{i_1}\cdots x_{i_m}\cdots x_{i_{m+s}}]_{\Pi_i\psi_{i,m}}$ should give

$$\begin{aligned} 0 = (1/\Pi_i(x_{i_m}))&\left\{\begin{matrix} \Pi_i(x_{i_m})/(x_i-x_{i_{m+1}})\cdots(x_i-x_{i_{m+s}}) \\ +\Pi_i(x_{i_{m+1}})/(x_{i_{m+1}}-x_{i_m})(x_{i_{m+1}}-x_{i_{m+2}})\cdots(x_{i_{m+1}}-x_{i_{m+s}})+\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +\Pi_i(x_{i_{m+s}})/(x_{i_{m+s}}-x_{i_m})\cdots(x_{i_{m+s}}-x_{i_{m+s-1}}) \end{matrix}\right\} \\ &+\cdots+[x_{i_m}\cdots x_{i_{m+s-1}}]_{1/\Pi_i}\left\{\begin{matrix}\Pi_i(x_{i_{m+s-1}})/(x_{i_{m+s-1}}-x_{i_{m+s}}) \\ +\Pi_i(x_{i_{m+s}})/(x_{i_{m+s}}-x_{i_{m+s-1}})\end{matrix}\right\}+a_{i_m}^{(s)}\Pi_i(x_{i_{m+s}}). \end{aligned} \tag{7}$$

The quantities in braces in (7) are

$$[x_{i_m} \cdots x_{i_{m+s}}]_{\Pi_i}, \cdots, [x_{i_{m+s-1}} x_{i_{m+s}}]_{\Pi_i}.$$

We now employ the Popoviciu-Steffensen theorem for the mth divided difference of the product $f(x)\phi(x)$ in terms of the divided differences of $f(x)$ and $\phi(x)$ [3; 4]:[1]

$$[x_0 x_1 \cdots x_m]_{f\phi} = \sum_{i=0}^{m} [x_0 \cdots x_i]_f [x_i \cdots x_m]_\phi. \tag{8}$$

Applying (8) to the product $1 \equiv \{1/\Pi_i(x)\}\Pi_i(x)$ for the points $(x_{i_m}, x_{i_{m+1}}, \cdots, x_{i_{m+s}})$, we have

$$\begin{aligned} 0 = {} & (1/\Pi_i(x_{i_m}))[x_{i_m} \cdots x_{i_{m+s}}]_{\Pi_i} + \cdots \\ & + [x_{i_m} \cdots x_{i_{m+s-1}}]_{1/\Pi_i}[x_{i_{m+s-1}} x_{i_{m+s}}]_{\Pi_i} \\ & + [x_{i_m} \cdots x_{i_{m+s}}]_{1/\Pi_i}\Pi_i(x_{i_{m+s}}). \end{aligned} \tag{9}$$

Comparison of (9) with (7) shows $a_{i_m}^{(s)} = [x_{i_m} \cdots x_{i_{m+s}}]_{1/\Pi_i}$.

The uniqueness of $P(x)$ is apparent, since specification by f_i, $\mathfrak{D}f_i, \cdots, \mathfrak{D}^{r_i-1}f_i$, $i=1, \cdots, n$, is equivalent to specification by certain $\sum_{i=1}^{n} r_i$ functional values $f_i, f_{i_1}, \cdots, f_{i_{r_i-1}}$ at $x_i, x_{i_1}, \cdots, x_{i_{r_i-1}}$, for which the Lagrange interpolation polynomial of degree $\sum_{i=1}^{n} r_i - 1$ (a rearrangement of $P(x)$) is unique.

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General Dynamics/Astronautics

[1] Popoviciu published a statement of (8) without proof in his thesis [3], apparently unknown to Steffensen whose proof appeared 5 or 6 years later [4].