UPPER SEMICONTINUOUS DECOMPOSITIONS
OF THE n-SPHERE

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We consider conditions under which an upper semicontinuous decomposition has the decomposition space which is a topological n-sphere. A special emphasis is placed on the case in which the decomposition has only a countable number of nondegenerate elements. If M is an n-manifold with boundary, Bd M and Int M will be used to denote the usual boundary and the interior of M. The set theoretical boundary of a subset A in a space will be denoted by b(A) and the closure of A by Cl A.

1. I-sets in $S^n$. A subset F of the n-sphere $S^n$ is called an I-set if $S^n - F$ is homeomorphic to $S^n - A$ for some countable subset A of $S^n$.

**Lemma 1.** Let A be a countable set in n-space $E^n$, $n > 1$. Then $E^n - A$ is connected.

**Lemma 1'.** If X is a connected n-manifold with boundary, $n > 1$, $X - A$ is connected for any countable set A.

**The Proof of Lemmas 1 and 1'.** If A is dense in $E^n$, Lemma 1 follows from Proposition A, p. 44, of [5] and the general case can be proved using this special case. For the proof of Lemma 1', use Lemma 1.

**Lemma 2.** Each component of an I-set F in $S^n$, $n > 1$, is closed.

**Proof.** Suppose a component C of F is not closed. Let $x \in Cl C - C$. Let h be a homeomorphism of $S^n - F$ onto $S^n - A$ for some countable set A. Since A is countable, there exist n-cells $D_1, D_2, \cdots$ such that $Int D_i \supset D_{i+1}$, $Bd D_i = \emptyset$ for each i and the intersection of the $D_i$ is the point h(x). Let $U_i$ be the complementary domain of $S_i = h^{-1}(Bd D_i)$ containing x and $U'_i$ the other complementary domain. Then $(U_i - F) + (U'_i - F)$ is a disjoint sum equal to a disjoint sum $h^{-1}(Int D_i - A) + h^{-1}(S^n - A - D_i)$. Since the four sets are open in $S^n - S_i - F$ and the last two sets are connected by Lemma 1', it follows that $U_i - F = h^{-1}(Int D_i - A)$. Each $U_i$ contains $Cl U_{i+1}$ and the intersection U of the $U_i$ is a compact set contained in $F + x$ as the intersection of $D_i$ is h(x). Since C is connected, $U_i$ is open and $S_i$ is disjoint from F, each $U_i$ contains C. Since C is closed in F, it follows that $Cl C = C + x$. Let $y \in b(C) - x$. (It is impossible that $b(C) = x$.)

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Let $y_1, y_2, \ldots$ be a sequence of points in $S^n - F$ that converges to $y$. Let $h(y_1), h(y_2), \ldots$ be a converging subsequence of $h(y_i)$. Then $y' = \lim h(y_i)$ is on $A$. Let $E_i$ be $n$-cells such that $\operatorname{Int} E_i \supset E_{i+1}$, $\operatorname{Bd} E_i \cap A = \emptyset$ for each $i$ and the intersection of $E_i$ is $y'$. Let $V_i$ be the complementary domain of $S_i = h^{-1}(\operatorname{Bd} E_i)$ containing $y$. Then $V_i - F = h^{-1}(\operatorname{Int} E_i - A)$ as $V_i$ contains infinitely many $y_i$. Then each $C_i V_i$ must contain $C_i C$ for the same reason as before, but this is impossible as the intersection of $E_i$ is disjoint from $h(x)$.

Let $F$ be any subset whatsoever of $S^n$. By $G(F)$ we denote the collection of the components of $F$ and the points of $S^n - F$.

**Theorem 1.** Let $G$ be a decomposition of $S^n$, $n > 1$, having only a countable number of nondegenerate elements. In order that $G$ be an upper semicontinuous decomposition of $S^n$ having an $n$-sphere as decomposition space, it is necessary and sufficient that $G = G(F)$ for some $I$-set $F$ in $S^n$.

**Proof. The necessity.** Let $g$ be an arbitrary element of $G$ and $X$ and $f$ respectively denote the decomposition space and the quotient map of $G$. Let $D_1, D_2, \ldots$ be $n$-cells in $X$ such that $\operatorname{Bd} D_i$ do not meet the image under $f$ of any nondegenerate element of $G$ and the intersection of $D_i$ is $f(g)$. Then $g$ is the intersection of a decreasing sequence of compact connected sets $U_i$, where $U_i$ are the closures of the complementary domains of $f^{-1}(\operatorname{Bd} D_i)$ containing $g$. Hence $g$ is connected.

Then $G = G(F)$ for the sum $F$ of the nondegenerate elements of $G$.

**The sufficiency.** We first show that $G(F)$ is upper semicontinuous. Let $x \in S^n - F$. It is easy to see that $x$ has an arbitrarily small neighborhood which is the sum of elements of $G(F)$. (Recall the proof of Lemma 2.) Let $C$ be a component of $F$ and $h$ be a homeomorphism of $S^n - F$ onto $S^n - A$ for some countable subset $A$ of $S^n$. Let $x_1, x_2, \ldots$ be a sequence of points in $S^n - F$ that converges to a point $x \in C$. Then by an argument similar to one in the proof of Lemma 2, we see that $C$ is contained in the intersection of a decreasing sequence of compact connected sets $\operatorname{Int} V_i$, where $V_i$ is a complementary domain of an $(n-1)$-sphere disjoint from $F$, and the intersection of $\operatorname{Cl} V_i$ is contained in $F$. That the intersection in fact is $C$ follows from the fact that $C$ is a component of $F$ and the intersection is connected. Then a $V_i$ with a sufficiently large $i$ is a very close neighborhood of $C$ which is the sum of elements of $G(F)$.

We now define a map $f$ of $S^n$ onto itself as follows. For each point $x \in S^n - F$, we let $f(x) = h(x)$. For each component $C$ of $F$, let $y_1, y_2, \ldots$ be a sequence of points in $S^n - F$ that converges to a point $y \in C$. Let $h(y_1), h(y_2), \ldots$ be a subsequence of $h(y_i)$ that con-

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1 The existence of $x$ is assured by the closedness of $C$. (See Lemma 1.)
verges to a point $y'$. Then $y' \in A$ as $y_1$ do not have a limit point in $S^n - F$. Let $f(C) = y'$. The map $f$ is then well-defined. For if $y'$ depended on $y$, etc. and in fact had two $y'$'s, by an argument similar to the proof of Lemma 2, we would find $C$ contained in two disjoint neighborhoods. We can also prove that $f$ is onto and sends distinct elements of $G(F)$ into distinct points. To prove the continuity of $f$, let $D$ be an $n$-cell in $S^n$ such that $\text{Bd} D$ is disjoint from $A$. Then $f^{-1}(\text{Int } D)$ is a complementary domain of $f^{-1}(\text{Bd } D) = h^{-1}(\text{Bd } D)$ which is certainly open. Since each open set in $S^n$ is the sum of such $\text{Int } D$'s, $f$ is continuous.

**Corollary 1.** An $I$-set $F$ in $S^n$, $n > 1$, has only a countable number of components.

**Corollary 2.** Let $A$ and $B$ be two countable subsets of $S^n$, $n > 1$. Then there exists a homeomorphism $h$ of $S^n$ onto itself such that $h(A) = B$ if and only if there is a homeomorphism $h'$ of $S^n - A$ onto $S^n - B$. Moreover, $h$ can be taken as an extension of $h'$.

**Remark.** Counter-examples show that Lemmas 1, 1' and 2, Theorem 1 and Corollaries 1 and 2 are false for $n = 1$ and that in the statement of Corollary 2 the words “countable subsets” cannot be replaced by “arcs.”

**Theorem 2.** Let $G$ be an upper semicontinuous decomposition of $S^3$ having a countable number of nondegenerate elements. If the decomposition space $X$ is a topological 3-sphere, then each element of $G$ has an open 3-cell as complement.

**Proof.** Let $f$ be the quotient map of $G$. We denote by $F$ the sum of the nondegenerate elements of $G$. Let $g$ be an arbitrary element of $G$. Since $f(F)$ is countable, there exist 3-cells $D_1, D_2, \ldots$ in $X$ such that $\text{Int } D_i \supset D_{i+1}$, $\text{Bd } D_i : f(F) = \emptyset$ for each $i$ and the intersection of $D_i$ is $f(g)$. Let $U_i = f^{-1}(\text{Int } D_i)$. Then $U_i$ is the complementary domain of $S_i = f^{-1}(\text{Bd } D_i)$ containing $g$. By the approximation [2] of Bing and the lemma below there exists a polyhedral 2-sphere $S'_i$ separating $\text{Cl } U_{i+1}$ from $S^3 - U_{i-1}$ for each $i > 1$. Hence there exist polyhedral 3-cells $E_i$, $E_2$, $\ldots$ such that $\text{Int } E_i \supset E_{i+1}$ for each $i$ and the intersection of $E_i$ is $f$. A repeated application of Theorem 1 of [6] proves the theorem.

**Lemma 3.** Let $S$ be an $(n-1)$-sphere in $S^n$ separating two connected closed subsets $A$ and $B$ in $S^n - S$. Then there exists a positive number $\varepsilon$ such that if $h$ is an $\varepsilon$-homeomorphism of $S$ onto $h(S)$ in $S^n - A - B$ then $h(S)$ separates $A$ from $B$. 

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Proof. We first show that there is a positive number $\epsilon$ such that if $h$ is an $\epsilon$-homeomorphism mentioned in the lemma, then there is a homotopy connecting the inclusion map of $S$ into $S^n$ and the map $h$. Since a neighborhood of $S$ can be uniformly continuously imbedded in $E^n$, we may assume that $S$ is in $E^n$ and construct a homotopy involving a very close neighborhood of $S$. Let $D_1, D_2, \cdots, D_k$ be a finite number of convex $n$-cells such that $\sum \text{Int } D_i \supset S$. Let $T$ be a triangulation of $S$ so fine that each simplex $\Delta$ of $T$ lies in an $\text{Int } D_{i(\Delta)}$. Let $\epsilon$ be a positive number such that each $\epsilon$-neighborhood of each simplex $\Delta$ of $T$ lies in $\text{Int } D_{i(\Delta)}$. Let $h$ be an $\epsilon$-homeomorphism of $S$ onto $h(S)$. Notice that $h(\Delta)$ lies in $\text{Int } D_{i(\Delta)}$. Then it is easy to construct the desired homotopy within $\sum \text{Int } D_i$. Now we come back to $S^n$ and suppose that $\epsilon$ was chosen such that not only $h$ has the property mentioned at the beginning of the proof but also the homotopy takes place outside the sum of an open connected neighborhood $U$ of $A$ and an open connected neighborhood $V$ of $B$. Before proceeding any further, we refer the reader to [7, Definition 8.2 and Theorem 8.3, p. 266]. Let $M = S^n - U - V$. A cycle $\Gamma$ irreducibly carried by $S$ and a cycle $\gamma$ irreducibly carried by the sum of a point of $A$ and a point of $B$ are linked. Since $\Gamma$ is homologous to $h(\Gamma)$ in $M$, $h(\Gamma)$ and $\gamma$ are also linked. Since $A$ and $B$ are connected and disjoint from $h(S)$, $h(S)$ must separate $A$ from $B$.

Remarks. A result for $S^2$ that is stronger than Theorem 2 is well known. While it may be true that an analogous theorem holds in $S^n$, we have no argument to prove it.

Suppose $F$ is a subset of $S^3$ having a countable number of components of which each has an open 3-cell complement. In general, $F$ is not an $I$-set. However, we can prove

Theorem 3. Let $F$ be a subset of $S^3$ having only a countable number of components. Suppose $F$ is a $G$ set and $G(F)$ is upper semicontinuous. Then $F$ is an $I$-set if and only if each component of $F$ has the open 3-cell complement.

Proof. The "if" part follows from Theorem 1 and [1, Theorem 1]. The "only if" part follows from Theorems 1 and 2.

Theorem 4. Let $F$ be the sum of a countable number of disjoint tame arcs in $S^3$. Then $F$ is an $I$-set if and only if $G(F)$ is upper semicontinuous.

Proof. Theorem 1 and [1, Theorem 3].

2. Similar $n$-cells in $S^n$. Let $S^{n-1}$ denote the "equator" of $S^n$. A topological $(n-1)$-sphere $S$ in $S^n$ is called tame if there exists a homeo-

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morphism of $S^n$ onto itself that sends $S$ onto $S^{n-1}$. A result [3] of Brown implies that $S$ is tame if and only if $S$ has a product neighborhood, i.e., the image of a homeomorphism of $S^{n-1} \times [0, 1]$ into $S^n$ that sends $S^{n-1} \times 1/2$ onto $S$. If $S$ is tame, its product neighborhood can be taken in any given neighborhood of $S$. An $n$-cell $D$ in $S^n$ is called tame if $\text{Bd } D$ is tame. Let $R_i$, $i = 1, 2$, be the subsets of $E^n$ consisting of the points $x$ with $\|x\| \leq i$. Two $n$-cells in $S^n$ are called concentric if there is a homeomorphism of one onto $R_3$ that sends the other onto $R_1$. If $D_1$ and $D_2$ are concentric, we also say that $D_1$ is concentric to $D_2$ and vice versa. Two $n$-cells $D$ and $D'$ with $D \subseteq D'$ are concentric if and only if $D' - \text{Int } D$ is homeomorphic to $S^{n-1} \times [0, 1]$. A finite number of disjoint $n$-cells $D_1, \ldots, D_k$ in $S^n$ are called similar if there is an $n$-cell to which each $D_i$ is concentric. The following problem should be important.

**The Concentricity Problem.** Let $D'$ be a tame $n$-cell in $S^n$ containing another tame $n$-cell $D$ in its interior. Are $D$ and $D'$ concentric?

This problem is equivalent to the following:

**The Similarity Problem.** Let $D_1, \ldots, D_k$ be disjoint tame $n$-cells in $S^n$. Are they similar?

These problems have the affirmative answers for $n \leq 3$. They are open for $n \geq 4$. We will require in certain theorems or lemmas that certain $n$-cells be similar or concentric. Such requirements may be deleted if and when either of the above problems is solved in the affirmative.

Let $D$ be an $n$-cell in $S^n$. An $n$-cell $D'$ in $S^n$ is called an $\epsilon$-pseudo-approximation of $D$ if $\text{Int } D'$ is an $\epsilon$-neighborhood of $D$.

**Lemma 4.** Let $D$ and $E$ be two tame $n$-cells in $S^n$ such that $D \cap E$. Then given a positive number $\epsilon$ there exists an $\epsilon$-pseudo-approximation $D'$ of $D$ such that $D'$ is concentric to $E$.

**Lemma 5.** Let $x_1, x_2, \ldots, x_k$ be distinct points in the interior of an $n$-cell $D$. Let $y_1, y_2, \ldots, y_k$ be distinct points in $\text{Int } D$. Then there exists a homeomorphism of $D$ onto itself such that it leaves $\text{Bd } D$ pointwise fixed and sends each $x_i$ to $y_i$.

The proofs of the above two lemmas are omitted.

**Theorem 5.** Let $D_1, \ldots, D_k$ be disjoint $n$-cells contained in, and concentric to, an $n$-cell $D$. Let $D'_1, \ldots, D'_k$ be disjoint $n$-cells contained in, and concentric to, $D$. Then there exists a homeomorphism $h$ of $D$ onto itself that moves no point of $\text{Bd } D$ and takes $D_i$ onto $D'_i$.

**Proof.** Let $x_i \in \text{Int } D_i$ and $y_i \in \text{Int } D'_i$. By Lemma 5, there exists a homeomorphism $h_i$ of $D$ onto itself that leaves each point of $\text{Bd } D$
fixed and takes $x_i$ to $y_i$. Let $h_0$ be a homeomorphism of $D$ onto itself that leaves $\text{Bd} D$ pointwise fixed and pulls $h_1(D_i)$ toward $y_i$ such that $h_2h_1(D_i) \subseteq \text{Int} D'_i$. Now $h_2h_1(D_i)$ is concentric to $D'_i$ for each $i$. Therefore there exists a homeomorphism $h_3$ that moves no point that is far from $\sum D'_i$ and takes $h_2h_1(D_i)$ onto $D'_i$. We finally let $h = h_3h_2h_1$.

**Corollary.** Let $D_1, D_2, \cdots, D_k$ be disjoint $n$-cells contained in, and concentric to, an $n$-cell $D$. Let $E_1, E_2, \cdots, E_k$ be disjoint $n$-cells contained in, and concentric to, an $n$-cell $E$. Then for any given homeomorphism $h$ of $\text{Bd} D$ onto $\text{Bd} E$, there exists an extension homeomorphism $H$ of $h$ that takes $D$ onto $E$ and each $D_i$ onto $E_i$.

**Theorem 6.** Let $G$ be an upper semicontinuous decomposition of $S^n$, $n > 1$, having a countable number of nondegenerate elements whose sum $F$ is closed. Then the decomposition space $X$ of $G$ is an $n$-sphere if and only if each element of $G$ has an open $n$-cell complement.

**Proof.** Suppose $X$ is an $n$-sphere. We denote the quotient map of $G$ by $f$. For given $g \in G$, there exist $n$-cells $D_1, D_2, \cdots$ in $X$ such that each $D_i$ is tame, $\text{Int} D_i \supseteq D_{i+1}$, $\text{Bd} D_i \cap f(E) = \emptyset$ and the intersection of $D_i$ is the point $f(g)$. Since each $(n-1)$-sphere $f^{-1}(\text{Bd} D_i)$ is apart from $F$ by a positive distance, it is tame. A repeated application of Lemma 4 shows that $S^n - g$ is an open $n$-cell.

The converse follows, since $F$ is a $G_2$ set, from [1, Theorem 1]. (Actually, Bing proves his theorem for $n = 3$ but the proof works for $n > 3$ as well. Also, Theorem 6 can be slightly strengthened. In fact, we only need to assume that $F + A$ is closed for some countable set $A$.)

**Theorem 7.** Let $G$ be a monotone upper semicontinuous decomposition of $S^n$ and $F$ be the sum of the nondegenerate elements of $G$. Suppose for each positive number $e$, there exist a finite number of disjoint $n$-cells $D_1, D_2, \cdots, D_k$ such that $\text{Cl} F \subseteq \sum \text{Int} D_i$ and each $D_i$ lies in an $e$-neighborhood of an element of $G$ it contains. Then the decomposition space $X$ is a topological $n$-sphere.

**Proof.** Let $D_1, \cdots, D_k$ be disjoint tame $n$-cells such that $\text{Cl} F \subseteq \sum \text{Int} D_i$ and each $D_i$ lies in a 1-neighborhood of an element of $G$ it contains. Let $e_1$ be a positive number smaller than $1/2$ such that no $2e_1$-neighborhood of any $\text{Bd} D_i$ meets $\text{Cl} F$. Let $D'_1, \cdots, D'_k$ be disjoint $n$-cells such that $\text{Cl} F \subseteq \sum \text{Int} D'_i$ and each $D'_i$ lies in an $e_1$-neighborhood of an element of $G$ it contains. Then $\sum D'_i \subseteq \sum \text{Int} D_i$. In general, there exist disjoint $n$-cells $D^{(p)}_1, \cdots, D^{(p)}_p$ such that $\text{Cl} F \subseteq \sum \text{Int} D^{(p)}_i$, each $D^{(p)}_i$ lies in a $1/2^p$-neighborhood of an element of $G$ it contains and $\sum D^{(p)}_i \subseteq \sum \text{Int} D^{(p-1)}_i$. 

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If \( g \) is an element of \( G \) contained in \( \text{Cl} F \), there exists a sequence \( D_{1p}, D_{2p}, \ldots, D^{(p)}_p, \ldots \) such that each \( D^{(p)}_p \) contains \( g \). Let \( g_p \subset D^{(p)}_p \) be an element of \( G \) of which \( D^{(p)}_p \) is in a \( 1/2^p \)-neighborhood. Let \( g' \in G \) be a limit set of \( g_p \)'s. Then an \( \epsilon \)-neighborhood \( U \) of \( g' \) meets infinitely many \( g_p \)'s. Therefore, by the upper semicontinuity of \( G \), \( U \) contains infinitely many \( g_p \). Hence, infinitely many \( D^{(p)}_p \) lie in \( \langle e + 1/2^p \rangle \)-neighborhoods of \( g' \). This means that the intersection of \( D^{(p)}_p \) is \( g' \) and \( g' = g \). On the other hand, if \( x \notin \text{Cl} F \), there exists \( p \) such that \( x \notin \sum D^{(p)}_p \).

We now simplify the situation a little. By shrinking \( D^{(p)} \)'s slightly inward, we may assume they are tame. Then using pseudo-approximations of Lemma 4, we may assume that \( D_i \) are similar and \( D_i(p) \) is concentric to \( D_{i-1}(p) \) if the latter contains the former. Using Corollary to Theorem 5 and the method of Harrold [4], we see that \( X \) is an \( n \)-sphere.

References

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