UPPER SEMICONTINUOUS DECOMPOSITIONS OF THE n-SPHERE

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We consider conditions under which an upper semicontinuous decomposition has the decomposition space which is a topological n-sphere. A special emphasis is placed on the case in which the decomposition has only a countable number of nondegenerate elements. If M is an n-manifold with boundary, Bd M and Int M will be used to denote the usual boundary and the interior of M. The set theoretical boundary of a subset A in a space will be denoted by $b(A)$ and the closure of A by Cl A.

1. **I-sets in S^n.** A subset F of the n-sphere S^n is called an *I-set* if S^n-F is homeomorphic to S^n-A for some countable subset A of S^n.

LEMMA 1. *Let A be a countable set in n-space E^n, $n>1$. Then E^n-A is connected.*

LEMMA 1′. *If X is a connected n-manifold with boundary, $n>1$, $X-A$ is connected for any countable set A.*

THE PROOF OF LEMMAS 1 AND 1′. If A is dense in E^n, Lemma 1 follows from Proposition A, p. 44, of [5] and the general case can be proved using this special case. For the proof of Lemma 1′, use Lemma 1.

LEMMA 2. *Each component of an I-set F in S^n, $n>1$, is closed.*

PROOF. Suppose a component C of F is not closed. Let $x\in \mathrm{Cl}\, C-C$. Let h be a homeomorphism of S^n-F onto S^n-A for some countable set A. Since A is countable, there exist n-cells $D_1, D_2, \cdots$ such that $\mathrm{Int}\, D_i\supset D_{i+1}$, $\mathrm{Bd}\, D_i\cdot A=\emptyset$ for each i and the intersection of the D_i is the point $h(x)$. Let U_i be the complementary domain of $S_i=h^{-1}(\mathrm{Bd}\, D_i)$ containing x and U_i' the other complementary domain. Then $(U_i-F)+(U_i'-F)$ is a disjoint sum equal to a disjoint sum $h^{-1}(\mathrm{Int}\, D_i-A)+h^{-1}(S^n-A-D_i)$. Since the four sets are open in S^n-S_i-F and the last two sets are connected by Lemma 1′, it follows that $U_i-F=h^{-1}(\mathrm{Int}\, D_i-A)$. Each U_i contains Cl U_{i+1} and the intersection U of the U_i is a compact set contained in $F+x$ as the intersection of D_i is $h(x)$. Since C is connected, U_i is open and S_i is disjoint from F, each U_i contains C. Since C is closed in F, it follows that $\mathrm{Cl}\, C=C+x$. Let $y\in b(C)-x$. (It is impossible that $b(C)=x$.)

Received by the editors March 16, 1961.

Let $y_1, y_2, \cdots$ be a sequence of points in $S^n - F$ that converges to y. Let $h(y_{i_1}), h(y_{i_2}), \cdots$ be a converging subsequence of $h(y_i)$. Then $y' = \lim h(y_{i_p})$ is on A. Let E_i be n-cells such that $\operatorname{Int} E_i \supset E_{i+1}$, $\operatorname{Bd} E_i \cdot A = \emptyset$ for each i and the intersection of E_i is y'. Let V_i be the complementary domain of $S_i' = h^{-1}(\operatorname{Bd} E_i)$ containing y. Then $V_i - F = h^{-1}(\operatorname{Int} E_i - A)$ as V_i contains infinitely many y_j. Then each $\operatorname{Cl} V_i$ must contain $\operatorname{Cl} C$ for the same reason as before, but this is impossible as the intersection of E_i is disjoint from $h(x)$.

Let F be any subset whatsoever of S^n. By $G(F)$ we denote the collection of the components of F and the points of $S^n - F$.

THEOREM 1. *Let G be a decomposition of S^n, $n > 1$, having only a countable number of nondegenerate elements. In order that G be an upper semicontinuous decomposition of S^n having an n-sphere as decomposition space, it is necessary and sufficient that $G = G(F)$ for some I-set F in S^n.*

PROOF. *The necessity.* Let g be an arbitrary element of G and X and f respectively denote the decomposition space and the quotient map of G. Let $D_1, D_2, \cdots$ be n-cells in X such that $\operatorname{Bd} D_i$ do not meet the image under f of any nondegenerate element of G and the intersection of D_i is $f(g)$. Then g is the intersection of a decreasing sequence of compact connected sets U_i, where U_i are the closures of the complementary domains of $f^{-1}(\operatorname{Bd} D_i)$ containing g. Hence g is connected. Then $G = G(F)$ for the sum F of the nondegenerate elements of G.

The sufficiency. We first show that $G(F)$ is upper semicontinuous. Let $x \in S^n - F$. It is easy to see that x has an arbitrarily small neighborhood which is the sum of elements of $G(F)$. (Recall the proof of Lemma 2.) Let C be a component of F and h be a homeomorphism of $S^n - F$ onto $S^n - A$ for some countable subset A of S^n. Let $x_1, x_2, \cdots$ be a sequence of points in $S^n - F$ that converges to a point $x \in C$.[1] Then by an argument similar to one in the proof of Lemma 2, we see that C is contained in the intersection of a decreasing sequence of compact connected sets $\operatorname{Cl} V_i$, where V_i is a complementary domain of an $(n-1)$-sphere disjoint from F, and the intersection of $\operatorname{Cl} V_i$ is contained in F. That the intersection in fact is C follows from the fact that C is a component of F and the intersection is connected. Then a V_i with a sufficiently large i is a very close neighborhood of C which is the sum of elements of $G(F)$.

We now define a map f of S^n onto itself as follows. For each point $x \in S^n - F$, we let $f(x) = h(x)$. For each component C of F, let $y_1, y_2, \cdots$ be a sequence of points in $S^n - F$ that converges to a point $y \in C$. Let $h(y_{i_1}), h(y_{i_2}), \cdots$ be a subsequence of $h(y_i)$ that con-

[1] The existence of x is assured by the closedness of C. (See Lemma 1.)

verges to a point y'. Then $y' \in A$ as y_i do not have a limit point in $S^n - F$. Let $f(C) = y'$. The map f is then well-defined. For if y' depended on y, etc. and in fact had two y''s, by an argument similar to the proof of Lemma 2, we would find C contained in two disjoint neighborhoods. We can also prove that f is onto and sends distinct elements of $G(F)$ into distinct points. To prove the continuity of f, let D be an n-cell in S^n such that Bd D is disjoint from A. Then $f^{-1}(\text{Int } D)$ is a complementary domain of $f^{-1}(\text{Bd } D) = h^{-1}(\text{Bd } D)$ which is certainly open. Since each open set in S^n is the sum of such Int D's, f is continuous.

COROLLARY 1. *An I-set F in S^n, $n > 1$, has only a countable number of components.*

COROLLARY 2. *Let A and B be two countable subsets of S^n, $n > 1$. Then there exists a homeomorphism h of S^n onto itself such that $h(A) = B$ if and only if there is a homeomorphism h' of $S^n - A$ onto $S^n - B$. Moreover, h can be taken as an extension of h'.*

REMARK. Counter-examples show that Lemmas 1, 1′ and 2, Theorem 1 and Corollaries 1 and 2 are false for $n = 1$ and that in the statement of Corollary 2 the words "countable subsets" cannot be replaced by "arcs."

THEOREM 2. *Let G be an upper semicontinuous decomposition of S^3 having a countable number of nondegenerate elements. If the decomposition space X is a topological 3-sphere, then each element of G has an open 3-cell as complement.*

PROOF. Let f be the quotient map of G. We denote by F the sum of the nondegenerate elements of G. Let g be an arbitrary element of G. Since $f(F)$ is countable, there exist 3-cells $D_1, D_2, \cdots$ in X such that Int $D_i \supset D_{i+1}$, Bd $D_i \cdot f(F) = \emptyset$ for each i and the intersection of D_i is $f(g)$. Let $U_i = f^{-1}(\text{Int } D_i)$. Then U_i is the complementary domain of $S_i = f^{-1}(\text{Bd } D_i)$ containing g. By the approximation [2] of Bing and the lemma below there exists a polyhedral 2-sphere S_i' separating Cl U_{i+1} from $S^3 - U_{i-1}$ for each $i > 1$. Hence there exist polyhedral 3-cells $E_1, E_2, \cdots$ such that Int $E_i \supset E_{i+1}$ for each i and the intersection of E_i is f. A repeated application of Theorem 1 of [6] proves the theorem.

LEMMA 3. *Let S be an $(n-1)$-sphere in S^n separating two connected closed subsets A and B in $S^n - S$. Then there exists a positive number ϵ such that if h is an ϵ-homeomorphism of S onto $h(S)$ in $S^n - A - B$ then $h(S)$ separates A from B.*

PROOF. We first show that there is a positive number ϵ such that if h is an ϵ-homeomorphism mentioned in the lemma, then there is a homotopy connecting the inclusion map of S into S^n and the map h. Since a neighborhood of S can be uniformly continuously imbedded in E^n, we may assume that S is in E^n and construct a homotopy involving a very close neighborhood of S. Let $D_1, D_2, \cdots, D_k$ be a finite number of convex n-cells such that $\sum \operatorname{Int} D_i \supset S$. Let T be a triangulation of S so fine that each simplex Δ of T lies in an $\operatorname{Int} D_{i(\Delta)}$. Let ϵ be a positive number such that each ϵ-neighborhood of each simplex Δ of T lies in $\operatorname{Int} D_{i(\Delta)}$. Let h be an ϵ-homeomorphism of S onto $h(S)$. Notice that $h(\Delta)$ lies in $\operatorname{Int} D_{i(\Delta)}$. Then it is easy to construct the desired homotopy within $\sum \operatorname{Int} D_i$. Now we come back to S^n and suppose that ϵ was chosen such that not only h has the property mentioned at the beginning of the proof but also the homotopy takes place outside the sum of an open connected neighborhood U of A and an open connected neighborhood V of B. Before proceeding any further, we refer the reader to [**7**, Definition 8.2 and Theorem 8.3, p. 266]. Let $M = S^n - U - V$. A cycle Γ irreducibly carried by S and a cycle γ irreducibly carried by the sum of a point of A and a point of B are linked. Since Γ is homologous to $h(\Gamma)$ in M, $h(\Gamma)$ and γ are also linked. Since A and B are connected and disjoint from $h(S)$, $h(S)$ must separate A from B.

REMARKS. A result for S^2 that is stronger than Theorem 2 is well known. While it may be true that an analogous theorem holds in S^n, we have no argument to prove it.

Suppose F is a subset of S^3 having a countable number of components of which each has an open 3-cell complement. In general, F is not an I-set. However, we can prove

THEOREM 3. *Let F be a subset of S^3 having only a countable number of components. Suppose F is a G set and $G(F)$ is upper semicontinuous. Then F is an I-set if and only if each component of F has the open 3-cell complement.*

PROOF. The "if" part follows from Theorem 1 and [**1**, Theorem 1]. The "only if" part follows from Theorems 1 and 2.

THEOREM 4. *Let F be the sum of a countable number of disjoint tame arcs in S^3. Then F is an I-set if and only if $G(F)$ is upper semicontinuous.*

PROOF. Theorem 1 and [**1**, Theorem 3].

2. **Similar n-cells in S^n.** Let S^{n-1} denote the "equator" of S^n. A topological $(n-1)$-sphere S in S^n is called *tame* if there exists a homeo-

morphism of S^n onto itself that sends S onto S^{n-1}. A result [3] of Brown implies that S is tame if and only if S has a product neighborhood, i.e., the image of a homeomorphism of $S^{n-1}\times[0, 1]$ into S^n that sends $S^{n-1}\times 1/2$ onto S. If S is tame, its product neighborhood can be taken in any given neighborhood of S. An n-cell D in S^n is called *tame* if Bd D is tame. Let R_i, $i=1, 2$, be the subsets of E^n consisting of the points x with $\|x\|\leqq i$. Two n-cells in S^n are called *concentric* if there is a homeomorphism of one onto R_2 that sends the other onto R_1. If D_1 and D_2 are concentric, we also say that D_1 is concentric to D_2 and vice versa. Two n-cells D and D' with $D\subset D'$ are concentric if and only if $D'-\text{Int } D$ is homeomorphic to $S^{n-1}\times[0, 1]$. A finite number of disjoint n-cells $D_1, \cdots, D_k$ in S^n are called *similar* if there is an n-cell to which each D_i is concentric. The following problem should be important.

THE CONCENTRICITY PROBLEM. Let D' be a tame n-cell in S^n containing another tame n-cell D in its interior. Are D and D' concentric?

This problem is equivalent to the following:

THE SIMILARITY PROBLEM. Let $D_1, \cdots, D_k$ be disjoint tame n-cells in S^n. Are they similar?

These problems have the affirmative answers for $n\leqq 3$. They are open for $n\geqq 4$. We will require in certain theorems or lemmas that certain n-cells be similar or concentric. Such requirements may be deleted if and when either of the above problems is solved in the affirmative.

Let D be an n-cell in S^n. An n-cell D' in S^n is called an ϵ-pseudo-approximation of D if Int D' is an ϵ-neighborhood of D.

LEMMA 4. *Let D and E be two tame n-cells in S^n such that D* Int E. *Then given a positive number ϵ there exists an ϵ-pseudo-approximation D' of D such that D' is concentric to E.*

LEMMA 5. *Let $x_1, x_2, \cdots, x_k$ be distinct points in the interior of an n-cell D. Let $y_1, y_2, \cdots, y_k$ be distinct points in* Int D. *Then there exists a homeomorphism of D onto itself such that it leaves* Bd D *pointwise fixed and sends each x_i to y_i.*

The proofs of the above two lemmas are omitted.

THEOREM 5. *Let $D_1, \cdots, D_k$ be disjoint n-cells contained in, and concentric to, an n-cell D. Let $D_1', \cdots, D_k'$ be disjoint n-cells contained in, and concentric to, D. Then there exists a homeomorphism h of D onto itself that moves no point of* Bd D *and takes D_i onto D_i'.*

PROOF. Let $x_i\in$Int D_i and $y_i\in$Int D_i'. By Lemma 5, there exists a homeomorphism h_1 of D onto itself that leaves each point of Bd D

fixed and takes x_i to y_i. Let h_2 be a homeomorphism of D onto itself that leaves Bd D pointwise fixed and pulls $h_1(D_i)$ toward y_i such that $h_2h_1(D_i) \subset \text{Int } D_i'$. Now $h_2h_1(D_i)$ is concentric to D_i' for each i. Therefore there exists a homeomorphism h_3 that moves no point that is far from $\sum D_i'$ and takes $h_2h_1(D_i)$ onto D_i'. We finally let $h = h_3h_2h_1$.

COROLLARY. *Let $D_1, D_2, \cdots, D_k$ be disjoint n-cells contained in, and concentric to, an n-cell D. Let $E_1, E_2, \cdots, E_k$ be disjoint n-cells contained in, and concentric to, an n-cell E. Then for any given homeomorphism h of* Bd D *onto* Bd E, *there exists an extension homeomorphism H of h that takes D onto E and each D_i onto E_i.*

THEOREM 6. *Let G be an upper semicontinuous decomposition of S^n, $n > 1$, having a countable number of nondegenerate elements whose sum F is closed. Then the decomposition space X of G is an n-sphere if and only if each element of G has an open n-cell complement.*

PROOF. Suppose X is an n-sphere. We denote the quotient map of G by f. For given $g \in G$, there exist n-cells $D_1, D_2, \cdots$ in X such that each D_i is tame, Int $D_i \supset D_{i+1}$, Bd $D_i \cdot f(F) = \emptyset$ and the intersection of D_i is the point $f(g)$. Since each $(n-1)$-sphere $f^{-1}(\text{Bd } D_i)$ is apart from F by a positive distance, it is tame. A repeated application of Lemma 4 shows that $S^n - g$ is an open n-cell.

The converse follows, since F is a G_δ set, from [1, Theorem 1]. (Actually, Bing proves his theorem for $n = 3$ but the proof works for $n > 3$ as well. Also, Theorem 6 can be slightly strengthened. In fact, we only need to assume that $F + A$ is closed for some countable set A.)

THEOREM 7. *Let G be a monotone upper semicontinuous decomposition of S^n and F be the sum of the nondegenerate elements of G. Suppose for each positive number ϵ, there exist a finite number of disjoint n-cells $D_1, D_2, \cdots, D_k$ such that* Cl $F \subset \sum$ Int D_i *and each D_i lies in an ϵ-neighborhood of an element of G it contains. Then the decomposition space X is a topological n-sphere.*

PROOF. Let $D_1, \cdots, D_k$ be disjoint tame n-cells such that Cl $F \subset \sum$ Int D_i and each D_i lies in a 1-neighborhood of an element of G it contains. Let e_1 be a positive number smaller than $1/2$ such that no $2e_1$-neighborhood of any Bd D_i meets Cl F. Let $D_1', \cdots, D_{k_1}'$ be disjoint n-cells such that Cl $F \subset \sum$ Int D_i' and each D_i' lies in an e_1-neighborhood of an element of G it contains. Then $\sum D_i' \subset \sum$ Int D_i. In general, there exist disjoint n-cells $D_1^{(p)}, \cdots, D_{k_p}^{(p)}$ such that Cl $F \subset \sum$ Int $D_i^{(p)}$, each $D_i^{(p)}$ lies in a $1/2^p$-neighborhood of an element of G it contains and $\sum D_i^{(p)} \subset \sum$ Int $D_i^{(p-1)}$.

If g is an element of G contained in Cl F, there exists a sequence $D_{i_0}, D'_{i_1}, \cdots, D^{(p)}_{i_p}, \cdots$ such that each $D^{(p)}_{i_p}$ contains g. Let $g_p \subset D^{(p)}_{i_p}$ be an element of G of which $D^{(p)}_{i_p}$ is in a $1/2^p$-neighborhood. Let $g' \in G$ be a limit set of g_p's. Then an e-neighborhood U of g' meets infinitely many g_p. Therefore, by the upper semicontinuity of G, U contains infinitely many g_p. Hence, infinitely many $D^{(p)}_{i_p}$ lie in $(e+1/2^p)$-neighborhoods of g'. This means that the intersection of $D^{(p)}_{i_p}$ is g' and $g'=g$. On the other hand, if $x \notin \text{Cl } F$, there exists p such that $x \notin \sum D^{(p)}_i$.

We now simplify the situation a little. By shrinking $D^{(p)}$'s slightly inward, we may assume they are tame. Then using pseudo-approximations of Lemma 4, we may assume that D_i are similar and $D_i(p)$ is concentric to $D^{(p-1)}_j$ if the latter contains the former. Using Corollary to Theorem 5 and the method of Harrold [4], we see that X is an n-sphere.

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