INJECTIVE MORPHISMS OF REAL ALGEBRAIC VARIETIES
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We need some preliminaries. Take the complex numbers as universal domain and define a real algebraic set to be the set of real points of an algebraic set that is defined over the reals. A real algebraic set $V$ is Zariski-dense in its Zariski closure $\overline{V}$, which is an algebraic set defined over the reals, and $V$ is the set of real points of $\overline{V}$. Any algebraic subset of $\overline{V}$ meets $V$ in a real algebraic subset. $V$ is irreducible (in its Zariski topology), i.e. $V$ is a real algebraic variety, if and only if $\overline{V}$ is irreducible (over the complex numbers), we define $\dim V = \dim \overline{V}$, and we call $P \in V$ simple if $P$ is a simple point of $\overline{V}$; such points $P$ exist, and there exist uniformizing parameters that are defined over the reals for $\overline{V}$ at such a point $P$, hence real local power series expansions, so that at each of its simple points $V$, in its ordinary topology, is locally a real analytic manifold of dimension $\dim V$.

For real algebraic sets $V$, $W$ define a rational map $f: V \to W$ to be the restriction to $V \times W$ of a rational map $f: V \to W$ that is defined over the reals; the rational map $f: V \to W$ is a morphism if $\overline{f}$ is defined at each point of $V$. Supposing the morphism of real algebraic varieties $f: V \to W$ to be such that $f(V)$ is Zariski-dense in $W$, a simple point $P \in V$ may be found such that $f(P)$ is simple on $W$ and $df$ has the correct rank $\dim V - \dim W$ at $P$, implying that, for the real analytic structure of $V$, $W$ at $P$, $f(P)$ respectively, $f$ is locally a projection onto a direct factor; in particular, if $f$ is finite-to-one, then $\dim V = \dim W$ and $\overline{f}$ has a finite degree $N$.

Specific examples of real algebraic sets: The real part of a complete nonsingular algebraic curve that is defined over the reals is the disjoint union of a finite number of circles, each with a real analytic structure. If $V$ is a real algebraic set of dimension one then a Zariski-open subset of $V$ is isomorphic to a Zariski-open subset of a nonsingular projective model of $V$ over the reals, hence the ordinary topology of $V$ is obtained by taking the topological sum of a finite number of points and circles, effecting a finite number of identifications, and deleting a finite number of points. If $V$ is a real hypersurface in $\mathbb{R}^n$,

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then V is the zero locus of a single real polynomial in n variables, say $f \in \mathbb{R}[X_1, \ldots, X_n]$; since V has simple points and rank $df = 1$ at these, f takes on both positive and negative values in $\mathbb{R}^n$—thus V separates $\mathbb{R}^n$, in the ordinary topology.

2. The following is the main point of Newman's proof.

If $f: V \to W$ is an injective morphism of real algebraic sets and $f(V)$ is Zariski-dense in W, then $\dim V = \dim W$ and $f(V)$ contains a Zariski-dense Zariski-open subset of W.

To prove this, we may take V, W irreducible, in which case we already know that $\dim V = \dim W$ and that f has finite degree N. Thus there exists a Zariski-closed proper subset $F \subset W$ such that $f^{-1}(Q)$ consists of precisely N points whenever $Q \in W - F$. For Q real, $f^{-1}(Q)$ consists of real points plus pairs of complex conjugate points, so taking $Q \in f(V) - F$ we deduce that N is odd and taking $Q \in W - F$ that $f^{-1}(Q)$ has at least one real point. Thus $f^{-1}(Q)$ is not empty if $Q \in W - W \cap F$.

Note that if V, W are real algebraic varieties and the above conditions hold, then f has odd degree. Conversely, if we delete the assumption of injectiveness and merely assume the degree of f odd, then the same conclusion follows.

3. We now prove the generalization of Newman's result.

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an injective morphism (of real algebraic sets, e.g. a real polynomial map), then f is also surjective.

First, injectivity and continuity in the ordinary topology imply that f is a homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^n - X$, where $X \subset \mathbb{R}^n$ is closed in the ordinary topology (invariance of domain). Consider the real algebraic sets $V_1 = X \cap \mathbb{R}^n$, $V_2 = f^{-1}(V_1)$, f is a homeomorphism (in the ordinary topology) between $\mathbb{R}^n - V_2$ and $\mathbb{R}^n - V_1$. If $V_3 \subset V_1$ is the Zariski-closure of $f(V_2)$, the previous section tells us that $\dim V_3 = \dim V_2$ and that $f(V_3)$ contains a Zariski-dense Zariski-open subset of $V_3$; in particular, $V_3 - f(V_3)$ is nowhere dense in $V_3$. Since $X = V_1 - f(V_2) = (V_1 - V_2) \cup (V_2 - f(V_2))$ and X is Zariski-dense in $V_1$, we must have $V_1 - V_2$ Zariski-dense in $V_1$, so that $\dim V_3 < \dim V_1$ unless $V_1$ is empty (which is to be proved). We are reduced to proving the following: if $V_1$, $V_2$ are (closed) real algebraic subsets of $\mathbb{R}^n$, then $\mathbb{R}^n - V_1$ and $\mathbb{R}^n - V_2$ are homeomorphic (in the ordinary topology) only if $\dim V_1 = \dim V_2$.

In proving the above statement we may assume $\dim V_1$, $\dim V_2 < n$. We wish first to remark that a quite elementary proof can be given for the case $n \leq 3$. As a matter of fact, we need only the following statements (each of which is either obvious or a direct consequence...
of the last paragraph of §1): If \( V \) is a proper real algebraic subset of \( \mathbb{R}^n \), where \( n \leq 3 \), then
(a) \( \mathbb{R}^n \) is contractible,
(b) \( \mathbb{R}^n - V \) is connected if and only if \( \dim V < n - 1 \),
(c) if \( n = 2 \) and \( \dim V = 0 \) then \( \mathbb{R}^n - V \) is not simply connected,
(d) if \( n = 3 \) and \( \dim V = 0 \) then \( \mathbb{R}^n - V \) is simply connected, but contains a 2-sphere not homotopic to a point,
(e) if \( n = 3 \) and \( \dim V = 1 \) then the one-point compactification of \( \mathbb{R}^n \) contains that of \( V \), which contains a topological circle, so \( \mathbb{R}^n - V \) is not simply connected.

To complete the proof in general we use the homology theory of Borel and Haefliger [1], with integers modulo two as coefficients. To each locally compact topological space \( X \) are associated \( \mathbb{Z}_2 \)-modules \( H_i(X) = H_i(X; \mathbb{Z}_2) \), one for each integer \( i \), vanishing for \( i > (\text{topological dimension of } X) \), reducing to the ordinary simplicial homology groups if \( X \) is a finite simplicial complex and to the usual relative homology groups if \( X \) is the complement of a subcomplex of a finite simplicial complex. There is an exact sequence relating the homologies of a space, a closed subspace and its complement, and, most essential, if \( X \) is a real algebraic set of dimension \( m \) there exists a (unique) fundamental class, i.e. an element of \( H_m(X) \) which induces at each simple point of \( X \) the generator of the local \( m \)-th homology group. This being so, consider for any proper real algebraic subset \( V \) of \( \mathbb{R}^n \) the exact sequence

\[
\cdots \rightarrow H_i(V) \rightarrow H_i(\mathbb{R}^n) \rightarrow H_i(\mathbb{R}^n - V) \rightarrow H_{i-1}(V) \rightarrow \cdots.
\]

We know that \( H_i(\mathbb{R}^n) \) is \( \mathbb{Z}_2 \) or 0, depending on whether \( i = n \) or \( i \neq n \), and \( H_i(V) = 0 \) if \( i > \dim V \), \( H_{\dim V}(V) \neq 0 \). Hence

\[
H_n(\mathbb{R}^n - V) \approx \mathbb{Z}_2 \oplus H_{n-1}(V),
\]

\[
H_i(\mathbb{R}^n - V) \approx H_{i-1}(V) \quad \text{if } i < n.
\]

Thus the knowledge of the groups \( \{H_i(\mathbb{R}^n - V)\} \) determines \( \dim V \), and we are done.

4. Here are some more, simpler, consequences of §2.

If \( f: V \rightarrow W \) is an injective morphism of complete nonsingular irreducible real algebraic curves, then \( f \) is bijective. For the proof of the result of §2 gives precise information in this case, since, taking \( V \) complete and nonsingular, as we may, over each point of \( W \) lies the same number of points of \( V \), counting multiplicities.

If \( f: V \rightarrow V \) is an injective morphism of the real algebraic curve \( V \) into itself, then \( f \) is bijective. This follows from the following more general
topological fact: If the topological space $V$ is the complement of a finite subset of a finite one-complex and if $f: V \to V$ is a continuous injection such that $V - f(V)$ consists of a finite number of points, then $f$ is a bijection. For the proof, note that the vertices of $V$ (i.e., points adherent to more than two ends of open line segments) are mapped by $f$ into vertices. Since the vertices are finite in number, $f$ is bijective on the vertices, and we may delete these from $V$. $V$ is now the topological sum of a finite number of disjoint points, line segments (open, half-open, or closed), and circles, and the proof here is straightforward.

If $f: G \to H$ is an injective homomorphism of real algebraic groups then $f(G)$ is a real algebraic subgroup of $H$. Using the Zariski topology, this reduces to showing that a dense abstract subgroup of a real algebraic group which contains a nonempty open subset is the whole group, which is a consequence of the fact that the closure of its complement is a proper closed subset invariant under a dense set of translations, therefore invariant under all translations, therefore empty.

5. We remark finally the following easy proof that if $f: k^n \to k^n$ is an injective polynomial map, where $k$ is any algebraically closed field, then $f$ is also surjective: Here $f$ must be birational or purely inseparable and a result of Chevalley [2, p. 195] shows that $f$ is open, so $f(k^n) = k^n - X$, with $X$ closed. Whether or not $f^{-1}$ is defined at a particular point depends on the poles of rational functions, so either $\dim X = n - 1$ or $X$ is empty. If $X$ were nonempty a nonconstant polynomial function on $k^n$ with zero locus contained in $X$ would give a similar function with no zero locus, which is impossible.

References


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