A THEOREM OF LEVITZKI

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Levitzki [2] proved that in a ring satisfying the ascending chain condition on left-ideals every left-ideal consisting of nilpotent elements is nilpotent. More recently, Goldie [1] has given a simpler proof. We present here a proof which is, in our opinion, even simpler and more elementary than that of Goldie. Moreover, it is completely self-contained.

Theorem (Levitzki). In a ring with ascending chain condition on left-ideals a nil left-ideal must be nilpotent.

Proof. We claim that we may assume that the ring $R$ has no non-zero nilpotent left-ideals, for if it did, it would have a nonzero nilpotent two-sided ideal, hence a maximal such, $N$. If the result were false, in $R/N$, which has no nilpotent left-ideals we would have a non-zero nil left-ideal.

So suppose that $R$ is a ring with ascending chain condition on left-ideals which has no nonzero nilpotent left-ideals and that $A \neq 0$ is a nil left-ideal of $R$. Let $a \neq 0 \in A$, and let $\mathfrak{M} = \{ ax \neq 0 | x \in R \}$. For $ax \in \mathfrak{M}$ let $L(ax) = \{ y \in R | yax = 0 \}$; these give us a set of left-ideals of $R$, which, by the ascending chain condition has maximal elements. We denote these maximal elements by $L(ax_i)$.

If $t \in R$ and $ax, \not\in \mathfrak{M}$ then $L(ax, t) \supset L(ax_i)$ which, together with $ax \not\in \mathfrak{M}$ forces $L(ax, t) = L(ax)$. We claim that given any finite number of such maximal $ax_1, \ldots, ax_n$—then there is an element $u \neq 0$ in $Rax_i R$ such that $ax_i u = \cdots = ax_n u = 0$. Suppose such a $u$ has been found such that $ax_1 u = \cdots = ax_{n-1} u = 0$. If $ax_n u = 0$ then we are done. So suppose any $u \neq 0 \in Rax_i R$ annihilating $ax_1, \ldots, ax_{n-1}$ does not annihilate $ax_n$. We claim there is such a $u \neq 0$ such that $uax_n \neq 0$; for if $uax_n = 0$ for all $u \in Rax_i R$ which annihilate $ax_1, \ldots, ax_{n-1}$, then since $uR$ also does the trick, $uRax_n = (0)$, whence $(Rax_n u)^2 = (0)$. But then $Rax_n u$ is a nilpotent left-ideal, so must be $(0)$. This forces $ax_n u = 0$ the desired result. Thus we may suppose that $uax_n \neq 0$.

Now $(uax_n)^t = 0$ for some $t$, since $x_n u a \in A$ is nilpotent. If $(uax_n)^t = 0$, $(uax_n)^{t-1} \neq 0$, then $uax_n(uax_n)^{t-1} = 0$. Since $u \in L(ax_n(uax_n)^{t-1})$ but $u \not\in L(ax_n)$, by the remark at the beginning of this paragraph, $ax_n(uax_n)^{t-1} = 0$. Thus $(uax_n)^{t-1} \neq 0$ annihilates $ax_1, \ldots, ax_{n-1}$ and $ax_n$; moreover since $u \in Rax_i R$, $(uax_n)^{t-1}$ also is.

Consider the ascending chain of left-ideals $Rax_1, Rax_1 + Rax_2, \ldots$,
A REMARK ON DIRECT PRODUCTS OF MODULES

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It is a standard result of abelian group theory that the direct product of an infinite number of infinite cyclic groups is not free [3, p. 48]. Various generalizations of this fact to modules over an arbitrary ring are contained in [1]. In this note we restrict our attention to integral domains, and in this setting present a generalization of the aforementioned result which is similar to, but in some sense stronger than, those of [1]. We then apply our theorem to show that any factor group $A$ of a direct product of infinite cyclic groups possesses the following amusing property: If $A$ is a subgroup of a direct sum of reduced torsion-free abelian groups, then it must be contained in the direct sum of a finite number of them.

Throughout this discussion, $R$ will be an integral domain which is not a field. All $R$-modules considered will be assumed to be unitary.

**Lemma.** Let $A$ be a torsion-free $R$-module, and set $A' = \bigcap_{\lambda \in R^*} \lambda A$. Then $A'$ is divisible. In particular, if $A$ is reduced (i.e., has no divisible submodules) then $\bigcap_{\lambda \in R^*} \lambda A = 0$.\(^1\)

\(^1\) $R^*$ means $R - \{0\}$.