ON TOTALLY MINIMAL SETS

HSIN CHU

The following theorem: "Let (X, T, π) be a transformation group, where T is the additive group of all real numbers with the usual topology and X is a compact Hausdorff space containing more than one point and let X be an almost periodic minimal orbit-closure under T. Then X is not totally minimal under T," which we may find in [**3**], is due to Professor E. E. Floyd. In this note, we show his theorem is still true if we weaken the condition that T is the real group to be any locally compact, non-totally-disconnected, abelian group and change the condition that X contains more than one point to be there is $a \in X$ such that aG_0 contains more than one point, where G_0 is the connected component of the identity of T. Thus we find a large class of transformation groups which is not totally minimal. The Pontrjagin duality of locally compact abelian groups is used in the proof. For the notations occurring here consult [**3**].

THEOREM. *Let (X, T, π) be a transformation group, where T is a locally compact, non-totally-disconnected, abelian group and X is a compact Hausdorff space and there is $a \in X$ such that aG_0 contains more than one point where G_0 is the connected component of identity in T, and let X be an almost periodic minimal orbit-closure under T. Then X is not totally minimal under T.*

PROOF. By a known result (see [**3**, p. 39]), X has a compact group structure such that a is the identity and $\pi_a: T \to X$ is a continuous homomorphism such that $\mathrm{Cl}((T)\pi_a) = \mathrm{Cl}(aT) = X$. It follows that $\mathrm{Cl}(aG_0)$ is a closed connected subgroup of X, which is not trivial. It is known there exists a continuous homomorhism f_a from $\mathrm{Cl}(aG_0)$ onto the circle group K and we can extend f_a to a continuous homomorphism $f: X \to K$ such that $f|\mathrm{Cl}(aG_0) = f_a$. Define $g: T \to K$ by $g = \pi_a f$. Consider $g|G_0: G_0 \to K$. Since $(G_0)\pi_a = aG_0$ is a nontrivial connected subgroup of X, which is dense in $\mathrm{Cl}(aG_0)$, $(G_0)g = (G_0\pi_a)f$ is a nontrivial connected subgroup K. However, the only nontrivial connected subgroup of the circle group K is K itself. It follows that $(G_0)g = K$. Hence $g: T \to K$ is an onto mapping. Let $g^{-1}(0) = L$ then L is a closed syndetic subgroup of T (see [**3**, 2.03]) and

$$0 \to L \xrightarrow{i} T \xrightarrow{g} K \to 0$$

is exact with i and g are relatively open mappings (with respect to their images), where i is the inclusion mapping. Thus, we have the

Received by the editors April 3, 1961.

following commutative diagram.

$$\begin{tikzcd}
 & & X \arrow[dr, "f"] & & \\
0 \arrow[r] & L \arrow[r, "i"'] \arrow[ur, "r"] & T \arrow[r, "g"'] \arrow[u, "\pi_a"'] & K \arrow[r] & 0
\end{tikzcd}$$

where $r = i\pi_a$. Then the following diagram

$$\begin{tikzcd}
 & & X^* \arrow[dl, "r^*"'] \arrow[d, "\pi_a{}^*"'] & & \\
0 & L^* \arrow[l] & T^* \arrow[l, "i^*"] & K^* \arrow[l, "g^*"] \arrow[ul, "f^*"'] & 0 \arrow[l]
\end{tikzcd}$$

is also commutative, where L^*, T^*, K^*, X^* are the character groups, with the usual compact-open topology, of L, T, K, X respectively. We know from [4], that $0 \to L^* \to_{i^*} T^* \to_{g^*} K^* \leftarrow 0$ is exact, where i^* and g^* are relatively open mappings (with respect to their images) and $K^* \cong Z$, where Z is the integer group with the discrete topology and f^* is isomorphic into such that f^* is relatively open with respect to its image and X^* is a nontrivial discrete group. From [1], we know $\pi_a^*: X^* \to T^*$ is continuously isomorphic into. Since $g^* i^*$ is trivial, r^* cannot be isomorphic into. By a result in [1], $r: L \to X$ is not dense, i.e. $\mathrm{Cl}((L)r) = \mathrm{Cl}(aL) \neq X$. This shows (X, L, π) is not a minimal set. This theorem is proved.

Using a similar argument, we have the following result:

COROLLARY. *Let (X, T, π) be a transformation group, where T is a locally compact abelian group and X is a compact, totally disconnected, Hausdorff space containing more than one point. If X is an almost periodic minimal orbit-closure under T. Then X is not totally minimal under T.*

REFERENCES

1. H. Anzai and S. Kakutani, *Bohr compactifications of a locally compact abelian group* I, Proc. Imperial Acad. Tokyo **19** (1943), 476–480.

2. ———, *Bohr compactifications of a locally compact abelian group.* II, Proc. Imperial Acad. Tokyo **19** (1943), 533–539.

3. W. Gottschalk and G. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ. Vol. 36, Amer. Math. Soc., Providence, R. I., 1955.

4. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Ind. Nos. 869–1145, Paris, Hermann, 1953.

NORTHWESTERN UNIVERSITY