ON TOTALLY MINIMAL SETS

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The following theorem: "Let \((X, T, \pi)\) be a transformation group, where \(T\) is the additive group of all real numbers with the usual topology and \(X\) is a compact Hausdorff space containing more than one point and let \(X\) be an almost periodic minimal orbit-closure under \(T\). Then \(X\) is not totally minimal under \(T\)," which we may find in [3], is due to Professor E. E. Floyd. In this note, we show his theorem is still true if we weaken the condition that \(T\) is the real group to be any locally compact, non-totally-disconnected, abelian group and change the condition that \(X\) contains more than one point to be there is \(a \in X\) such that \(aG_0\) contains more than one point, where \(G_0\) is the connected component of the identity of \(T\). Thus we find a large class of transformation groups which is not totally minimal. The Pontrjagin duality of locally compact abelian groups is used in the proof. For the notations occurring here consult [3].

**Theorem.** Let \((X, T, \pi)\) be a transformation group, where \(T\) is a locally compact, non-totally-disconnected, abelian group and \(X\) is a compact Hausdorff space and there is \(a \in X\) such that \(aG_0\) contains more than one point where \(G_0\) is the connected component of identity in \(T\), and let \(X\) be an almost periodic minimal orbit-closure under \(T\). Then \(X\) is not totally minimal under \(T\).

**Proof.** By a known result (see [3, p. 39]), \(X\) has a compact group structure such that \(a\) is the identity and \(\pi_a: T \rightarrow X\) is a continuous homomorphism such that \(\text{Cl}((T)\pi_a) = \text{Cl}(aT) = X\). It follows that \(\text{Cl}(aG_0)\) is a closed connected subgroup of \(X\), which is not trivial. It is known there exists a continuous homomorphism \(f_0\) from \(\text{Cl}(aG_0)\) onto the circle group \(K\) and we can extend \(f_0\) to a continuous homomorphism \(f: X \rightarrow K\) such that \(f|\text{Cl}(aG_0) = f_0\). Define \(g: T \rightarrow K\) by \(g = \pi_a f\). Consider \(g|G_0: G_0 \rightarrow K\). Since \((G_0)\pi_a = aG_0\) is a nontrivial connected subgroup of \(X\), which is dense in \(\text{Cl}(aG_0)\), \((G_0)g = (G_0)\pi_a f\) is a nontrivial connected subgroup \(K\). However, the only nontrivial connected subgroup of the circle group \(K\) is \(K\) itself. It follows that \((G_0)g = K\). Hence \(g: T \rightarrow K\) is an onto mapping. Let \(g^{-1}(0) = L\) then \(L\) is a closed syndetic subgroup of \(T\) (see [3, 2.03]) and

\[
0 \rightarrow L \xrightarrow{i} T \xrightarrow{g} K \rightarrow 0
\]

is exact with \(i\) and \(g\) are relatively open mappings (with respect to their images), where \(i\) is the inclusion mapping. Thus, we have the

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following commutative diagram.

\[
\begin{array}{c}
\text{0} \rightarrow L \rightarrow T \rightarrow K \rightarrow 0 \\
\end{array}
\]

where \( r = i \pi_a \). Then the following diagram

\[
\begin{array}{c}
\text{0} \leftarrow L^* \leftarrow T^* \leftarrow K^* \leftarrow 0 \\
\end{array}
\]

is also commutative, where \( L^*, T^*, K^*, X^* \) are the character groups, with the usual compact-open topology, of \( L, T, K, X \) respectively. We know from [4], that \( 0 \rightarrow L^* \rightarrow T^* \rightarrow K^* \leftarrow 0 \) is exact, where \( i^* \) and \( g^* \) are relatively open mappings (with respect to their images) and \( K^* \cong Z \), where \( Z \) is the integer group with the discrete topology and \( f^* \) is isomorphic into such that \( f^* \) is relatively open with respect to its image and \( X^* \) is a nontrivial discrete group. From [1], we know \( \pi_a^*: X^* \rightarrow T^* \) is continuously isomorphic into. Since \( g^*i^* \) is trivial, \( r^* \) cannot be isomorphic into. By a result in [1], \( r: L \rightarrow X \) is not dense, i.e. \( \text{Cl}((L)r) = \text{Cl}(aL) \neq X \). This shows \((X, L, \pi)\) is not a minimal set. This theorem is proved.

Using a similar argument, we have the following result:

**Corollary.** Let \((X, T, \pi)\) be a transformation group, where \( T \) is a locally compact abelian group and \( X \) is a compact, totally disconnected, Hausdorff space containing more than one point. If \( X \) is an almost periodic minimal orbit-closure under \( T \). Then \( X \) is not totally minimal under \( T \).

**References**