CONVERGENCE OF APPROXIMATING POLYNOMIALS

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I. The problem we wish to consider is the following. For each positive integer \(n\), let \(E_n\) be a finite subset of \([-1, 1]\) containing at least \(n\) points. For a real-valued continuous function \(f\) defined on \([-1, 1]\) let \(p_n(f, E_n)\) be the unique polynomial of degree at most \(n-1\) of best approximation in the Chebycheff sense to \(f\) on \(E_n\). Is it possible to choose a fixed sequence \(\{E_n\}\) so that for each \(f\), continuous on \([-1, 1]\), \(p_n(f, E_n)\) converges to \(f\) uniformly on \([-1, 1]\)?

A classical result of Faber [4] states that if, for each \(n\), \(E_n\) contains exactly \(n\) points, this choice is never possible. In this case, of course, \(p_n(f, E_n)\) is just the polynomial which interpolates to \(f\) at the points of \(E_n\).

In this paper we shall prove that the result of Faber still holds if each \(E_n\) contains no more than \(re + 1\) points. On the other hand, letting \(\|f\| = \sup_{-1 \leq t \leq 1} |f(t)|\), we obtain \(\|f - p_n(f, E_n)\| \to 0\) for each \(f\) continuous on \([-1, 1]\), if and only if there exists a constant \(K\) independent of \(n\), such that for each polynomial \(p_n(x)\) of degree at most \(n-1\), if \(|p_n(x)| \leq 1\) for each \(x \in E_n\), then \(\|p_n\| \leq K\).

The existence of such sets \(E_n\) was first proved by Bernstein [1, pp. 55–57]. In fact \(E_n = \{\cos ((k \pi / m)) \}, k = 0, 1, \cdots, m\), where \(m/n > \pi / 2 \cdot 2^{1/2}\) is a simple example. It is further shown in [1] that for each fixed \(\lambda > 1\) if \(k_n\) satisfies \(k_n/n > \lambda\) then we may choose a sequence \(\{E_n\}\) with the desired properties and such that the cardinality of \(E_n = k_n\). Namely, assuming \(k_n \leq 2n\), let \(E_n\) consist of the points \(\cos((2k - 1)/2n)\pi, k = 1, \cdots, n\), together with the points \(\cos((l\pi / n)\) where \(l\) is an integer satisfying \(k_n - n - 1 = [n/l]\) and \(t = 0, 1, \cdots, [n/l]\).

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II. Let \(C = C[-1, 1]\), the Banach space of real-valued continuous functions on \([-1, 1]\) provided with the norm \(\|f\| = \sup_{-1 \leq t \leq 1} |f(t)|\). Let \(H_n\) be the \(n\) dimensional sub-space of polynomials of degree \(n-1\). Denote by \(P_n\) the mapping \(f \mapsto p_n(f, E_n)\). \(P_n\) is a continuous mapping of \(C\) onto \(H_n\) satisfying \(P_n P_n = P_n\) for \(m \geq n\). In general, \(P_n\) is not linear, but if \(E_n\) contains either \(n\) or \(n+1\) points, then \(P_n\) is linear which is the crucial fact needed in the following:

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Theorem 1. For \( n = 1, 2, \ldots \) fix a sequence of finite subsets \( E_n \) of \([-1, 1]\). If each \( E_n \) contains either \( n \) or \( n+1 \) points, then there exists an \( f \in C \) to which \( p_n(f, E_n) \) fails to converge uniformly on \([-1, 1]\).

Proof. Assume for the moment that \( P_n \) is linear for each \( n \). Let \( f \in H_n \). Then if \( m > n \), \( P_m(f) = f \) since \( P_m \) maps \( C \) onto \( H_n \) and \( P_m P_n = P_n \). By the Weierstrass approximation theorem the polynomials are dense in \( C \), hence we may infer from the principle of uniform boundedness [3, Theorem II.3.6] that \( P_n(f) \to f \) for each \( f \in C \) iff \( \sup_n \| P_n \| < \infty \), where \( \| P_n \| = \sup_{x \in [-1,1]} \| P_n(f) \| \). But by [5, Hilfssatz 3, p. 495] if \( P_n \) is any bounded projection of \( C \) onto \( H_n \), \( \| P_n \| \geq \ln(n-1)/8\pi^{1/2} \).

Now \( P_n \) is clearly linear if \( E_n \) contains exactly \( n \) points. If \( E_n \) contains \( n+1 \) points \( x_i, -1 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 1 \), let \( q_{n+1}(x) = \sum_{k=0}^n a_k x^k \) and \( r_{n+1}(x) = \sum_{k=0}^n b_k x^k \) be the unique polynomials determined by the conditions \( q_{n+1}(x_i) = (-1)^i, r_{n+1}(x_i) = f(x_i), i = 1, 2, \ldots, n+1 \). It is easily seen by considering the determinants involved that \( a_n \neq 0 \). Therefore, let \( p_n(x) = r_{n+1}(x) - (b_n/a_n) q_{n+1}(x) \). The mapping \( f \to p_n \) is clearly linear, since \( f \to r_{n+1} \) and \( f \to b_n \) are both linear. But \( f(x_i) - p_n(x) = (b_n/a_n)(-1)^i, i = 1, 2, \ldots, n+1 \). Therefore, \( p_n = p_n(f, E_n) \) by the classical result of de la Vallée Poussin [2] which completes the proof.

We note two facts. First, it may be easily verified that if \( E_n \) contains more than \( n+1 \) points, \( P_n \) is never linear. Secondly, if \( q_n(f, E_n) \) denotes the polynomial of best approximation to \( f \) on \( E_n \) in the sense of least squares, then the same argument as above shows that if \( \{ E_n \} \) is any sequence of finite subsets of \([-1, 1]\) containing at least \( n \) points, then for some \( f \in C \), \( q_n(f, E_n) \) fails to converge uniformly to \( f \). This follows since the mapping \( f \to q_n(f, E_n) \) is always linear and idempotent.

III. We now prove the convergence criterion.

Theorem 2. For each \( n > 0 \) let \( E_n \) be a finite subset of \([-1, 1]\). Then \( \| f - p_n(f, E_n) \| \to 0 \) for each \( f \in C \) iff there exists a constant \( K \) such that if \( p \in H_n \), \( |p(x)| \leq 1, x \in E_n \), then \( \| p \| < K \).

Proof. This is a theorem of uniform boundedness type, and although the operators \( P_n \) are nonlinear the proof resembles that for the linear case.

With no loss in generality, we assume each \( E_n \) contains at least \( n+1 \) points. For fixed \( E_n \) and \( p \in H_n \) let \( \delta(p) = \sup_{x \in E_n} |f(x) - p(x)| \). Then by a well-known result of de la Vallée Poussin [2] \( p_n = p_n(f, E_n) \) is characterized uniquely by the condition that there exist \( n+1 \) points \( x_i \) in \( E_n \), \( x_i \leq x_{i+1} \), for which either
\[ f(x_i) - p_n(x_i) = (-1)^i \delta(p_n), \quad i = 1, 2, \ldots, n + 1, \]
or
\[ f(x_i) - p_n(x_i) = (-1)^{i+1} \delta(p_n), \quad i = 1, 2, \ldots, n + 1. \]

From this it follows easily that the operator \( P_n \) is homogeneous, and if \( q \) is a polynomial of degree \(< n\), then for each \( f \in C \), \( P_n(f + q) = P_n(f) + q \). Moreover \( E_n \) satisfies the condition of the theorem iff \( \sup_n \| P_n \| < \infty \). For, by the above remarks, if \( p \in H_n \) and \( \| p(x) \| \leq 1 \), \( x \in E_n \), then \( p = P_n(f, E_n) \) for some \( f \). \( \| f \| \leq 2 \). Conversely, if \( \| f \| \leq 1 \), then \( \| p_n(x) \| \leq 2 \) for \( x \in E_n \), for otherwise \( p(x) = 0 \) would provide a better approximation on \( E_n \). Therefore, suppose \( \sup_n \| P_n \| = K < \infty \). For each \( \varepsilon > 0 \) choose a polynomial \( q \) such that
\[ \| f - q \| < \varepsilon. \]
If \( n \) is the degree of \( q \) and \( n > n_\varepsilon \), then
\[ \| f - P_n(f) \| \leq \| f - q \| + \| q - P_n(f) \| \leq \| f - q \| + \| P_n(q - f) \| \leq \varepsilon(1 + K). \]

Therefore, \( \| f - P_n(f) \| \to 0 \) and the condition is sufficient.

Conversely, suppose \( \| f - P_n(f) \| \to 0 \) for each \( f \in C \). Since each \( P_n \) is continuous, \( S_n,k = \{ f \in C : \| P_n(f) \| \leq k \} \) is a closed subset of \( C \). Therefore, by the Baire category theorem, for some \( k > 0 \), \( S_k = \bigcap_{n=1}^{\infty} S_n,k \) contains an open set. Consequently, there exists a polynomial \( q(x) \) and a positive number \( \delta \) such that if \( \| f \| < \delta \), then \( f + q \in S_k \).

Hence, for \( n > \) degree of \( q \) and \( \| f \| < \delta \),
\[ \| P_n(f) \| \leq \| P_n(q) \| + \| P_n(f - q) \| \leq \| q \| + k. \]

Using this and the continuity of each \( P_n \) it follows that
\[ \sup_n \| P_n(f) \| < \infty, \quad \| f \| < \delta, \]
and the theorem is proved.

**Bibliography**


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