CONVERGENCE OF APPROXIMATING POLYNOMIALS

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I. The problem we wish to consider is the following. For each positive integer \( n \), let \( E_n \) be a finite subset of \([-1, 1]\) containing at least \( n \) points. For a real-valued continuous function \( f \) defined on \([-1, 1]\) let \( p_n(f, E_n) \) be the unique polynomial of degree at most \( n - 1 \) of best approximation in the Chebycheff sense to \( f \) on \( E_n \). Is it possible to choose a fixed sequence \( \{E_n\} \) so that for each \( f \), continuous on \([-1, 1]\), \( p_n(f, E_n) \) converges to \( f \) uniformly on \([-1, 1]\)?

A classical result of Faber [4] states that if, for each \( n \), \( E_n \) contains exactly \( n \) points, this choice is never possible. In this case, of course, \( p_n(f, E_n) \) is just the polynomial which interpolates to \( f \) at the points of \( E_n \).

In this paper we shall prove that the result of Faber still holds if each \( E_n \) contains no more than \( n + 1 \) points. On the other hand, letting \( \|f\| = \sup_{-1 \leq t \leq 1} |f(t)| \), we obtain \( \|f - p_n(f, E_n)\| \to 0 \) for each \( f \) continuous on \([-1, 1]\), if and only if there exists a constant \( K \) independent of \( n \), such that for each polynomial \( p_n(x) \) of degree at most \( n - 1 \), if \( |p_n(x)| \leq 1 \) for each \( x \in E_n \), then \( \|p_n\| \leq K \).

The existence of such sets \( E_n \) was first proved by Bernstein [1, pp. 55–57]. In fact \( E_n = \{\cos(k\pi/m)\}, \ k = 0, 1, \cdots, m \), where \( m/n > \pi/2 \cdot 2^{1/2} \) is a simple example. It is further shown in [1] that for each fixed \( \lambda > 1 \) if \( k_n \) satisfies \( k_n/n > \lambda \) then we may choose a sequence \( \{E_n\} \) with the desired properties and such that the cardinality of \( E_n = k_n \). Namely, assuming \( k_n \leq 2n \), let \( E_n \) consist of the points \( \cos((2k - 1)/2n)\pi, \ k = 1, \cdots, n \), together with the points \( \cos(l\pi/n) \) where \( l \) is an integer satisfying \( k_n - n - 1 = [n/l] \) and \( t = 0, 1, \cdots, \frac{n}{l} \).

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II. Let \( C = C[-1, 1] \), the Banach space of real-valued continuous functions on \([-1, 1]\) provided with the norm \( \|f\| = \sup_{-1 \leq t \leq 1} |f(t)| \). Let \( H_n \) be the \( n \) dimensional sub-space of polynomials of degree \( n - 1 \). Denote by \( P_n \) the mapping \( f \to p_n(f, E_n) \). \( P_n \) is a continuous mapping of \( C \) onto \( H_n \) satisfying \( P_m P_n = P_n \) for \( m \geq n \). In general, \( P_n \) is not linear, but if \( E_n \) contains either \( n \) or \( n + 1 \) points, then \( P_n \) is linear which is the crucial fact needed in the following:

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Theorem 1. For \( n = 1, 2, \ldots \) fix a sequence of finite subsets \( E_n \) of \([-1, 1]\). If each \( E_n \) contains either \( n \) or \( n+1 \) points, then there exists an \( f \in C \) to which \( p_n(f, E_n) \) fails to converge uniformly on \([-1, 1]\).

Proof. Assume for the moment that \( p_n \) is linear for each \( n \). Let \( f \in H_n \). Then if \( m > n \), \( p_m(f) = f \) since \( p_n \) maps \( C \) onto \( H_n \) and \( p_m p_n = p_n \). By the Weierstrass approximation theorem the polynomials are dense in \( C \), hence we may infer from the principle of uniform boundedness [3, Theorem II.3.6] that \( p_n(f) \to f \) for each \( f \in C \) iff \( \sup_n \| p_n \| < \infty \), where \( \| p_n \| = \sup_{x \in E_1} \| p_n(f) \| \). But by [5, Hilfssatz 3, p. 495] if \( p_n \) is any bounded projection of \( C \) onto \( H_n \), \( \| p_n \| \leq \ln(n-1)/8\pi^{1/2} \).

Now \( p_n \) is clearly linear if \( E_n \) contains exactly \( n \) points. If \( E_n \) contains \( n+1 \) points \( x_i, -1 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 1 \), let \( q_{n+1}(x) = \sum_{k=0}^{n} a_k x^k \) and \( r_{n+1}(x) = \sum_{k=0}^{n} b_k x^k \) be the unique polynomials determined by the conditions \( q_{n+1}(x_i) = (-1)^i, r_{n+1}(x_i) = f(x_i), i = 1, 2, \ldots, n + 1 \). It is easily seen by considering the determinants involved that \( a_n \neq 0 \). Therefore, let \( p_n(x) = r_{n+1}(x) - (b_n/a_n) q_{n+1}(x) \). The mapping \( f \to p_n \) is clearly linear, since \( f \to r_{n+1} \) and \( f \to b_n \) are both linear. But \( f(x_i) - p_n(x_i) = (b_n/a_n)(-1)^i, i = 1, 2, \ldots, n + 1 \). Therefore, \( p_n = p_n(f, E_n) \) by the classical result of de la Vallée Poussin [2] which completes the proof.

We note two facts. First, it may be easily verified that if \( E_n \) contains more than \( n+1 \) points, \( p_n \) is never linear. Secondly, if \( q_n(f, E_n) \) denotes the polynomial of best approximation to \( f \) on \( E_n \) in the sense of least squares, then the same argument as above shows that if \( \{ E_n \} \) is any sequence of finite subsets of \([-1, 1]\) containing at least \( n \) points, then for some \( f \in C \), \( q_n(f, E_n) \) fails to converge uniformly to \( f \). This follows since the mapping \( f \to q_n(f, E_n) \) is always linear and idempotent.

III. We now prove the convergence criterion.

Theorem 2. For each \( n > 0 \) let \( E_n \) be a finite subset of \([-1, 1]\). Then \( \| f - p_n(f, E_n) \| \to 0 \) for each \( f \in C \) iff there exists a constant \( K \) such that if \( p \in H_n \), \( \| p \| \leq 1, x \in E_n \), then \( \| p \| < K \).

Proof. This is a theorem of uniform boundedness type, and although the operators \( p_n \) are nonlinear the proof resembles that for the linear case.

With no loss in generality, we assume each \( E_n \) contains at least \( n+1 \) points. For fixed \( E_n \) and \( p \in H_n \) let \( \delta(p) = \sup_{x \in E_n} |f(x) - p(x)| \). Then by a well-known result of de la Vallée Poussin [2] \( p_n = p_n(f, E_n) \) is characterized uniquely by the condition that there exist \( n+1 \) points \( x_i \) in \( E_n \), \( x_i \leq x_{i+1} \), for which either
From this it follows easily that the operator $P_n$ is homogeneous, and if $q$ is a polynomial of degree $< n$, then for each $f \in C$, $P_n(f+q) = P_n(f)+P_n(q) = P_n(f) + q$. Moreover $E_n$ satisfies the condition of the theorem iff $\sup_n \| P_n \| < \infty$. For, by the above remarks, if $p \in H_n$ and $|p(x)| \leq 1$, $x \in E_n$, then $p = P_n(f, E_n)$ for some $f$, $\| f \| \leq 2$. Conversely, if $\| f \| \leq 1$, then $|p_n(x)| \leq 2$ for $x \in E_n$ for otherwise $p(x) = 0$ would provide a better approximation on $E_n$. Therefore, suppose $\sup_n \| P_n \| = K < \infty$. For each $\varepsilon > 0$ choose a polynomial $q$ such that $\| f - q \| < \varepsilon$. If $n_0$ is the degree of $q$ and $n > n_0$, then

$$\| f - P_n(f) \| \leq \| f - q \| + \| q - P_n(f) \| \leq \varepsilon(1 + K).$$

Therefore, $\| f - P_n(f) \| \to 0$ and the condition is sufficient.

Conversely, suppose $\| f - P_n(f) \| \to 0$ for each $f \in C$. Since each $P_n$ is continuous, $S_{n,k} = \{ f \in C : \| P_n(f) \| \leq k \}$ is a closed subset of $C$. Therefore, by the Baire category theorem, for some $k > 0$, $S_k = \cap_{n=1}^{\infty} S_{n,k}$ contains an open set. Consequently, there exists a polynomial $g(x)$ and a positive number $\delta$ such that if $\| f \| < \delta$, then $f + g \in S_k$. Hence, for $n > \deg q$ and $\| f \| < \delta$,

$$\| P_n(f) \| \leq \| P_n(q) \| + \| P_n(f - q) \| \leq \| q \| + k.$$ 

Using this and the continuity of each $P_n$ it follows that

$$\sup_n \| P_n(f) \| < \infty, \quad \| f \| < \delta,$$

and the theorem is proved.

Bibliography