A DECOMPOSITION THEOREM FOR
n-DIMENSIONAL MANIFOLDS

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Throughout our discussion an \( n \)-dimensional manifold will mean a connected, separable metric space in which each point has an open \( n \)-cell neighborhood. Our main result can be stated in the following manner.

**Theorem 1.** Let \( M^n \) be an \( n \)-dimensional manifold. Then \( M^n = P^n \cup C \), where \( P^n \) is homeomorphic to euclidean \( n \)-space, \( E^n \), and \( C \) is a closed subset of \( M^n \) of dimension at most \( n-1 \); and \( P^n \cap C = \emptyset \).

Considered from one point of view Theorem 1 is a generalization of Corollary 1 in [3]. From still another the result says that any \( n \)-manifold is "almost triangulable." The proof of Theorem 1 leads to more interesting results in the case of compact manifolds which we shall consider presently.

The steps in the proof will be described here. If \( C^n \) is a closed \( n \)-cell in \( M^n \) such that \( \text{Bd} C^n \), the boundary of \( C^n \), is bicollared in \( M^n \), [2], and if \( \{ a_i \} \) is a countable dense subset of \( M^n \setminus C^n \), consider the set \( C^n \cup \bigcup_i a_i \). Does this set lie on the interior of an \( n \)-cell in \( M^n \) with a bicollared boundary? If this were the case and if \( C_i \) is such an \( n \)-cell, one could ask if \( C_i \cup a_i \) lies interior to an \( n \)-cell in \( M^n \) with a bicollared boundary. Continuing in this way with sets of the form \( C_i \cup a_{i+1} \), if such enclosure is always possible, we obtain an increasing sequence \( \{ C_i \} \) of closed \( n \)-cells in \( M^n \), \( \text{Bd} C_i \) is bicollared in \( M^n \) and \( \text{int} C_{i+1} \supseteq C_i \), where interior \( C_{i+1} \) is written \( \text{int} C_{i+1} \). Next we observe that \( P^n = \bigcup_i C_i \) is \( E^n \) by either a direct construction of cells with annuli between them or by applying the main result of [1]. Then \( M^n - P^n = C \) is nowhere dense in \( M^n \) and closed since \( P^n \) is open. The sets \( P^n \) and \( C \) would then meet the requirements of Theorem 1.

From this outline it is clear that the proof of Theorem 1 follows immediately from a lemma.

**Lemma 1.** Let \( M^n \) be an \( n \)-manifold and \( D^n \) a closed \( n \)-cell in \( M^n \) with bicollared boundary. Then if \( p \) is any point in \( M^n \), \( D^n \cup p \) lies in \( \text{int} D_i^n \), where \( D_i^n \) is a closed \( n \)-cell and \( \text{Bd} D_i^n \) is bicollared.

**Proof.** Let \( q \) be any point in \( \text{int} D^n \). There is a homeomorphism \( h \) of \( M^n \) onto \( M^n \) which is pointwise fixed outside any neighborhood \( V \) of \( D^n \) and which carries \( D^n \) into any preassigned neighborhood \( \mathcal{U} \) of \( q \).

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while $h(q) = q$. This follows from the fact that $\text{Bd} \, D^n$ is bicollared.

Since $M^n$ is a manifold, the set $p \cup q$ lies interior to an $n$-cell in $M^n$ and so interior to an $n$-cell with a bicollared boundary in $M^n$. Evidently we need only select $u$ in the interior of such a cell to obtain the proof of Lemma 1.

With the proof of Lemma 1 we obtain Theorem 1. In the case $M^n$ is compact the conditions on $C$ are stronger.

**Theorem 2.** Let $M^n$ be a compact $n$-dimensional manifold. Then $M^n = P^n \cup C$, where $P^n$ is homeomorphic to $E^n$, and $C$ is a nonseparating continuum in $M^n$; $P^n \cap C = \emptyset$.

It is convenient to call the decomposition $P^n \cup C$ of $M^n$ in the above theorem as a standard decomposition if $P^n$ is obtained as in the proof of Theorem 1.

**Corollary 1.** Let $M^n$ be a compact $n$-manifold and $M^n = P^n \cup C$ a standard decomposition of $M^n$. Then if there is a homeomorphism $h$ of $M^n$ onto $M^n$ such that $h(C) \subset P^n$, then $M^n$ is an $n$-sphere.

**Proof.** By Theorem 2, $C$ is compact and so $h(C)$ lies in the interior of a closed $n$-cell $C'$ in $P^n$. By the construction of $P^n$, $M^n$ is the union of two closed $n$-cells with no boundary points in common. Whence, as in Lemma 3 of [4], one can conclude that $M^n$ is a sphere.

**Corollary 2.** Let $M^n$ be a compact $n$-compact $n$-manifold and let $M^n = P^n_1 \cup C_1 = P^n_2 \cup C_2$ be two standard decompositions. If $C_1 \cap C_2 = \emptyset$, then $M^n$ is a sphere.

It should be pointed out that the set $C$ in a standard decomposition need not be nice. In the case of the 2-sphere $S^2$, $C$ may be any nonseparating 1-dimensional continuum in $S^2$; so $C$ need not be locally connected.

**Theorem 3.** Let $M^n$ be an $n$-manifold and $S^n$, the $n$-sphere. Then there is a map $f$ from $M^n$ onto $S^n$ such that each point of $S^n$ has a degenerate inverse except perhaps for one point $p$, and $\dim f^{-1}(p) \leq n - 1$.

**Proof.** The representation of $P^n$ as an increasing sequence of $n$-cells provides an evident map of the type described with $C$ as the only possible nondegenerate inverse. In case $M^n = E^n$, $C$ may be void; however, one may arrange it so that $C$ is not void even in this case.

In the proof of Corollary 1 to Theorem 2 we observed that a compact $n$-manifold which fails to be a sphere cannot be the union of two closed $n$-cells having no boundary points in common. Similar results
may be obtained for open regions. If $M^n$ is a compact $n$-manifold, $P^n \cup C = M^n$, a standard decomposition, let $C$ be in an open set $U$ in $M^n$ such that $U$ is homeomorphic to a subset of $E^n$. Then if $h$ is an imbedding of $U$ in $E^n$ we note that $h(C)$ is the limit in $E^n$ of a strictly decreasing sequence of closed $n$-cells with bicollared boundaries in $h(U) \subset E^n$. Whence, we obtain $M^n$ as a union of closed $n$-cells with disjoint bicollared boundaries and so $M^n$ is an $n$-sphere. We can then assert another theorem.

**Theorem 4.** If $M^n$ is a compact $n$-manifold which is not an $n$-sphere and if $M^n = P^n \cup C$ is a standard decomposition, then $C$ has no neighborhood in $M^n$ which can be imbedded in $E^n$.

**References**


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