TRANSLATION-INVARIANT FUNCTIONALS

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1. Introduction. Suppose \( X \) is a translation-invariant linear subspace of \( C_0(R) \) (the space of all continuous functions on the real line \( R \) that vanish at infinity) that is dense in \( C_0(R) \) with respect to the uniform topology. If \( \mu \) is a measure on the line such that

\[
\int_{-\infty}^{\infty} f(x + t) d\mu(x) = \int_{-\infty}^{\infty} f(x) d\mu(x)
\]

for all \( f \in X \) and all \( t \in R \), does it follow that \( \mu \) is a constant multiple of the Lebesgue measure?

Our interest in this question arose in the following context. Let \( \Gamma \) be the dual group of a locally compact abelian group \( G \) (written additively), and let \((x, \gamma)\) be the value of the character \( \gamma \in \Gamma \) at the point \( x \in G \). If \( f \in L^1(G) \), its Fourier transform is defined by

\[
\hat{f}(\gamma) = \int_G f(x)(-x, \gamma) dx \quad (\gamma \in \Gamma),
\]

where \( dx \) denotes the Haar measure of \( G \). The inversion formula

\[
f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) d\gamma \quad (x \in G),
\]

where \( d\gamma \) denotes the (suitably normalized) Haar measure of \( \Gamma \), is valid for all \( f \in \mathcal{P}^1 \), the space of all linear combinations of positive definite functions in \( L^1(G) \). In two standard texts \([1, p. 143; 2, p. 413]\), (3) is proved by first showing that there is a positive measure \( \mu \) on \( \Gamma \) such that

\[
f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) d\mu(\gamma)
\]

and

\[
\int_\Gamma \hat{f}(\gamma) d\mu(\gamma) = \int_\Gamma \hat{f}(\gamma + \gamma') d\mu(\gamma)
\]

for all \( \gamma' \in \Gamma \) and for all \( \hat{f} \in \mathcal{P}^1 \) (the set of all Fourier transforms of

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members of $P^1$. Since $P^1$ is dense in $C_0(\Gamma)$, it is concluded from (5) that $\mu$ is a Haar measure on $\Gamma$, and then (4) establishes (3).

In Theorem 1 below we show that the correctness of the italicized statement in the preceding sentence stems from the fact that $P^1$ is an algebra (under pointwise multiplication). This point is glossed over in both [1] and [2], and the reader is left with the erroneous impression that the only measures $\mu$ on $\Gamma$ that satisfy (5) for a dense subset of functions in $C_0(\Gamma)$ are the Haar measures. We are thus led to the following question, to which we have obtained partial answers:

Suppose $X$ is a translation-invariant subspace of $C_0(G)$, $\mu$ is a measure on $G$, and $\mu$ acts invariantly on $X$, i.e.,

$$\int_G f(x + t) d\mu(x) = \int_G f(x) d\mu(x) \quad (f \in X, t \in G).$$

What information does this give about $\mu$, and what information does it give about the translation-invariant functional $T_\mu$ defined on $X$ by

$$T_\mu(f) = \int_G f(x) d\mu(x).$$

By a measure we always mean a complex, countably additive, regular set function defined on the Borel sets of $G$ which is finite for all sets with compact closure. The space of all $f \in C_0(G)$ with compact support will be denoted by $C_c(G)$.

2. Uniqueness theorems.

**Theorem 1.** Suppose $A$ is a dense translation-invariant subalgebra of $C_0(G)$, $\mu$ is a measure on $G$, and $\int |f| d|\mu| < \infty$ for all $f \in A$. If $\mu$ acts invariantly on $A$, then $\mu$ is a constant (complex) multiple of the Haar measure of $G$.

**Proof.** Choose $g \in C_c(G)$. Since $A$ is dense in $C_0(G)$, $A$ contains a function $h$ which vanishes at no point of the support of $g$. Let $k = g/h$; then $k \in C_c(G)$, and so there is a sequence $\{f_n\}$ in $A$ that converges to $k$ uniformly on $G$. Since $\int |h| d|\mu| < \infty$, Lebesgue’s dominated convergence theorem shows that

$$\lim_{n \to \infty} \int_G f_n(x + t) h(x + t) d\mu(x) = \int_G g(x + t) d\mu(x)$$

for every $t \in G$. Since $f_n h \in A$, the left side of (8) is independent of $t$. The same is therefore true of the right side, and we have shown that

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\( \mu \) acts invariantly on \( C_c(G) \).

Since every measure on \( G \) is determined by its action on \( C_c(G) \), the uniqueness theorem for Haar measure\(^3\) completes the proof.

**Theorem 2.** Suppose \( \mu \) is a measure on \( G \) that acts invariantly on a translation-invariant linear subspace \( X \) of \( C_0(G) \), such that \( \int |f| d|\mu| < \infty \) and \( \int |f| dx < \infty \) for all \( f \in X \). If
\[
(9) \quad \hat{\mu}(0) \neq 0
\]
for some \( g \in X \), then there exists a constant \( \lambda \) such that
\[
(10) \quad \int \hat{\mu}(x) \, dx = \lambda \int \hat{\mu}(x) \, dx \quad (f \in X).
\]

**Proof.** For any \( f \in X \), we have
\[
\hat{\mu}(0) \int \hat{\mu}(x) \, dx = \int \hat{\mu}(0) \int \hat{\mu}(x) \, dx + \int \hat{\mu}(x) \, dx
\]
\[
= \int \hat{\mu}(0) \int \hat{\mu}(x) \, dx
\]
by the invariant action of \( \mu \) on \( X \) and by Fubini's theorem. Since \( \int \hat{\mu}(x) \, dx = (1 + e^x) dx \) is symmetric in \( f \) and \( g \), and so
\[
(12) \quad \hat{\mu}(0) \int \hat{\mu}(x) = \hat{\mu}(0) \int \hat{\mu}(x).
\]
This is (10), with \( \lambda = [\hat{\mu}(0)]^{-1} \int g d\mu \).

**Remarks.** (a) This proof is patterned after a simple uniqueness proof for Haar measure on abelian groups [1, p. 116].

(b) We did not assume that \( X \) is dense in \( C_0(G) \). (Cf. Theorem 3, however.)

(c) The conclusion of the theorem amounts to the statement that the functional \( T_\mu \) defined on \( X \) by (7) is also given by integration with respect to a Haar measure. That is to say, \( \mu \) acts on \( X \) like a Haar measure. *This does not imply, however, that \( \mu \) is itself translation-invariant.*

For example, let \( X \) be the set of all \( f \in C_c(R) \) such that \( \int e^{-x} f(x) e^x \, dx = 0 \), and take \( \mu(x) = (1 + e^x) \, dx \). The space \( X \) is translation-invariant, \( \mu \) acts invariantly on \( X \), and Theorem 3 below shows that \( X \) is even dense in \( C_0(R) \).

\(^3\) Uniqueness of translation-invariant measures, which is customarily stated only for positive measures, is valid for complex measures as well. For abelian groups this follows, for example, from Theorem 2 below, with \( X = C_c(G) \).
(d) If condition (9) is omitted from Theorem 2, the conclusion is no longer valid, even if $X$ is dense in $C_0(G)$ and if $\mu$ is a positive measure. To see this, let $X$ be the linear space generated by all translates of even functions $f \in C_c(R)$ for which $f(0) = 0$, and take $d\mu(x) = x^2 dx$. Note that

$$\int_{-\infty}^{\infty} f(x + t)x^2 dx = \int_{-\infty}^{\infty} f(x)(x^2 - 2tx + t^2) dx.$$ 

If $f$ is even, then $\int f(x)x dx = 0$. Since $\int f(x)dx = f(0) = 0$ for all $f \in X$, $\mu$ acts invariantly on $X$. But if $f(0) = 2, f(3) = -1, f(6) = 0$, $f$ is linear between these points, $f(x) = 0$ for $x > 6$, and $f$ is even, then $f \in X$ but $\int_{-\infty}^{\infty} x^2 f(x) dx < 0$. Thus $\int f d\mu$ is not a constant multiple of $f(0)$.

3. Subspaces of $C_0(R)$. In this section, we confine our attention to the group $R$ of real numbers. We show, first, that translation-invariant subspaces of $C_0(R)$ are usually dense.

**Theorem 3.** If a subspace $X$ of $C_0(R)$ contains all translates of some nonzero function $f$ with compact support, then $X$ is dense in $C_0(R)$.

**Proof.** Suppose $\mu$ is a bounded measure on $R$ that annihilates $X$; then

$$\int_{-\infty}^{\infty} f(x - t)d\mu(x) = 0 \quad (t \in R).$$

If $F$ is the Fourier transform of $\hat{f}$, where $\hat{f}(x) = f(-x)$, (13) implies that $F \cdot \hat{\mu} = 0$. But $F$ is an analytic function, hence has only isolated zeros, and since $\hat{\mu}$ is continuous, we conclude that $\hat{\mu} = 0$. By the uniqueness theorem for Fourier-Stieltjes transforms, $\mu = 0$, and so $X$ is dense in $C_0(R)$, by the Hahn-Banach theorem.

**Remark.** We did not really need to assume that $f \in C_c(R)$. All we needed was that the zeros of $\hat{f}$ should form a nowhere dense set.

The examples following Theorem 2 show that a measure that acts invariantly on a translation-invariant subspace $X$ of $C_c(R)$ need not be a constant multiple of Lebesgue measure. We have already remarked, however, that a translation-invariant functional (7) on $X$ is necessarily a multiple of the Lebesgue integral in case some member of $X$ has a nonzero integral. In the next theorem, we recapture the uniqueness property of translation-invariant functionals on $X$ even when every member of $X$ has zero integral. The proof is similar to that of Theorem 2.

**Theorem 4.** Suppose $\mu$ is a measure on $R$ that acts invariantly on a translation-invariant linear subspace $X$ of $C_c(R)$. Let $p$ be the smallest nonnegative integer such that
\( \int_{-\infty}^{\infty} x^p g(x) \, dx \neq 0 \)

for some \( g \in X \). Then there is a constant \( \lambda \) such that

\( \int_{-\infty}^{\infty} f(x) \, d\mu(x) = \lambda \int_{-\infty}^{\infty} x^p f(x) \, dx \quad (f \in X). \)

Note that the right side of (15) is a constant times the \( p \)th derivative of \( \tilde{f} \) at the origin.

**Proof.** The case \( p = 0 \) is dealt with in Theorem 2.

Suppose \( p > 0 \), \( f \in X \), \( f \neq 0 \), and the support of \( f \) is contained in the interval \([-A, A]\). Let \( f_0 = f \), and for \( k \geq 1 \), let \( f_k(x) = \int_{-A}^{x} f_{k-1}(t) \, dt \).

By induction, one obtains the well-known formula

\( f_k(x) = \frac{1}{(k-1)!} \int_{-A}^{x} (x-t)^{k-1} f(t) \, dt \quad (k = 1, 2, \ldots). \)

We have chosen \( p \) so that

\( \int_{-A}^{A} f_{k-1}(t) \, dt = f_k(A) = \frac{1}{(k-1)!} \int_{-A}^{A} (A-t)^{k-1} f(t) \, dt = 0 \)

for \( 1 \leq k \leq p \). Thus, \( f_0, f_1, \ldots, f_p \) all have compact support.

Furthermore,

\( f_p(0) = \int_{-\infty}^{\infty} f_p(t) \, dt = f_{p+1}(A) = \frac{1}{p!} \int_{-\infty}^{\infty} (A-t)^{p+1} f(t) \, dt \)

\( = \frac{(-1)^p}{p!} \int_{-\infty}^{\infty} p f(t) \, dt. \)

Hence, if \( g \) is a function in \( X \) that satisfies (14), then \( g_p(0) \neq 0 \).

Now, as in the proof of Theorem 2, for any \( f \in X \),

\( g_p(0) \int_{-\infty}^{\infty} f(x) \, d\mu(x) = \int_{-\infty}^{\infty} d\mu(x) \int_{-\infty}^{\infty} g_p(t) f(x-t) \, dt. \)

Integration by parts \( p \) times yields

\( \int_{-\infty}^{\infty} g_p(t) f(x-t) \, dt = \int_{-\infty}^{\infty} g(t) f_p(x-t) \, dt = \int_{-\infty}^{\infty} f_p(t) g(x-t) \, dt, \)

and comparison of (19) and (20) shows that

\( g_p(0) \int_{-\infty}^{\infty} f(x) \, d\mu(x) = f_p(0) \int_{-\infty}^{\infty} g(x) \, d\mu(x). \)
With $g$ fixed so that $g_p(0) \neq 0$, (21) together with (18) yields (15) and completes the proof.

By virtue of Theorem 4 every measure that acts invariantly on a translation-invariant subspace $X$ of $C_e(R)$ differs from a measure $\lambda x^\nu dx$, for suitable $\lambda$ and $\nu$, by a measure that vanishes on all of $X$. The theorem also furnishes some nonzero measures that vanish on $X$, namely, the measures $x^k dx$ for $0 \leq k < p$. Moreover, it provides a clue for finding still other such measures.

For every $f \in X$, the Fourier transform $\hat{f}$ of $f$ can be extended to an entire function in the complex plane. Associate with each complex number $\alpha$ a nonnegative integer $m(\alpha)$, the largest integer $k$ such that $\hat{f}(\alpha) (z - \alpha)^{-k}$ is regular at $z = \alpha$ for all $f \in X$, and let $E$ be the set of all $\alpha$ for which $m(\alpha) > 0$. If $X \neq \{0\}$, it is clear that $E$ has no limit point in the finite plane. An equivalent definition of $m(\alpha)$ is that

$$\int_{-\infty}^{\infty} x^k e^{-i\alpha x} f(x) dx = 0$$

for $0 \leq k < m(\alpha)$ and all $f \in X$; whereas

$$\int_{-\infty}^{\infty} x^{m(\alpha)} e^{-i\alpha x} g(x) dx \neq 0$$

for some $g \in X$. If now $\alpha \in E$ and

$$d\mu(x) = (c_0 + c_1 x + \cdots + c_k x^k) e^{-i\alpha x} dx \quad (k < m(\alpha)),$$

then (22) implies that

$$\int_{-\infty}^{\infty} f(x) d\mu(x) = 0$$

for all $f \in X$.

To sum up, any finite linear combination of measures (24) and the measure $x^{m(0)} dx$ acts invariantly on $X$, and so do certain infinite sums. For instance, if $\alpha_1, \alpha_2, \alpha_3, \cdots$ are points of $E$ and if $\{c_j\}$ tends to 0 rapidly enough, the series

$$s(x) = \sum_{j=1}^{\infty} c_j e^{-i\alpha_j x}$$

converges uniformly on compact subsets of $R$, and the measure $d\mu(x) = s(x) dx$ acts invariantly on $X$.

Both examples which follow Theorem 2 are of this sort. It is noteworthy that the complex zeros of the Fourier transforms play a role here.
We conclude with an example of a space $X \subset C_c(R)$ (a dense subspace of $C_0(R)$, by Theorem 3) which contains a nontrivial nonnegative function and on which a positive measure (not a Haar measure) acts invariantly: $X$ consists of all $f \in C_c(R)$ with $f(1) = f(-1) = 0$, and $d\mu(x) = (2 + \sin x)dx$. Since $2i \sin x = e^{ix} - e^{-ix}$, $\int_X f(x) \sin x \, dx = 0$ for all $f \in X$, and so $\mu$ acts invariantly on $X$; also, $X$ contains the nonnegative triangular function $f$ defined by

$$f(x) = \max (2\pi - |x|, 0) \quad (-\infty < x < \infty).$$

REFERENCES


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BARRELLED SPACES AND THE OPEN MAPPING THEOREM

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1. Introduction. If $E$ and $F$ are any two topological vector spaces then the following statement may or may not be true:

(A) If $f$ is any linear and continuous mapping of $E$ onto $F$ then $f$ is open.

It is well known [1] that (A) is true when $E$ and $F$ are Fréchet spaces. An extension due to Pták [6], and Robertson and Robertson [7] is that (A) is true if $E$ is $B$-complete and $F$ is barrelled ($t$-space). We ask here whether these results characterize Fréchet and $B$-complete spaces respectively. More precisely, let $\mathcal{F}$ and $\mathcal{J}$ denote the classes of all Fréchet and barrelled spaces respectively. We ask if a topological vector space $E$, having the property that (A) is true whenever $F \in \mathcal{J}$, is necessarily a Fréchet ($B$-complete) space.

A well-known example of an LF-space and a theorem of Dieudonné and Schwartz [5, Theorem 1] supplies a counterexample to the above for $\mathcal{F}$. Here, we give an example showing that the other case is also false.

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