

A NOTE ON TRANSLATION INVARIANTS

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1. Introduction. Autocorrelation functions play a central role in many engineering applications. The following operation describes a certain optical system [2]: from a function f of two variables, representing the transparency of a photographic slide, another transparency function F is produced on a photographic plate which is the autocorrelation function of f ; that is,

$$F(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) f(\xi - x, \eta - y) d\xi d\eta.$$

The essential feature of the function F is that it remains unchanged if f is replaced by f^* where $f^*(\xi, \eta) = f(\xi + \alpha, \eta + \beta)$. A uniqueness question now arises; if f and g have the same autocorrelation function, are they related by a rectilinear motion of the plane?

The answer to this question is negative as is shown in §4 by counterexample. In §2 we present this problem in a more general setting and in §3 the main theorem.

2. Notation and definitions. Let G be a locally compact abelian group, μ Haar measure, and $L_{1,r}(G)$ the space of real-valued μ -integrable functions on G . For each $s \in G$ define the translation operator $T_s: L_{1,r}(G) \rightarrow L_{1,r}(G)$ by $(T_s f)(x) = f(sx)$ and denote by $\mathfrak{X}(f)$ the set $\{g: g \in L_{1,r}(G), (T_s f)(x) = g(x) (\mu \text{ a.e.}) \text{ for some } s \in G\}$ of translates of f . A functional $\rho(\cdot)$ with domain $L_{1,r}(G)$ will be called *translation invariant* if $\rho(f) = \rho(g)$ for all $g \in \mathfrak{X}(f)$. A class of translation invariants $\mathfrak{P} = \{\rho_\omega(\cdot): \omega \in \Omega\}$ is *complete* if $\rho_\omega(f) = \rho_\omega(g)$ for all $\omega \in \Omega$ implies $g \in \mathfrak{X}(f)$.

The k th order autocorrelation function of $f \in L_{1,r}(G)$ is the real-valued function $\rho_k(f) \equiv \rho_k(f)(x_1, x_2, \dots, x_k)$ with domain $G^{(k)} = G \times G \times \dots \times G$ (k copies) defined formally by

$$\rho_k(f)(x_1, x_2, \dots, x_k) = \int_G f(\xi) f(\xi x_1) f(\xi x_2) \dots f(\xi x_k) \mu(d\xi).$$

The proof that $\rho_k(f)$ is $\mu^{(k)}$ -almost everywhere¹ defined and in $L_{1,r}(G)$ is a modification of the argument of [1, p. 121] used in establishing that the convolution of integrable functions is integrable. The map-

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¹ $\mu^{(k)}$ denotes the direct product measure on $G^{(k)}$ given by μ on G .

ping $f \rightarrow \rho_k(f)$ of $L_{1,r}(G)$ into $L_{1,r}(G^{(k)})$ is a translation invariant functional as μ is Haar measure.

Let \hat{G} denote the character group of G . For $h \in L_{1,r}(G^{(k)})$ define the " k -dimensional" Fourier transform by

$$\hat{h}(\chi_1, \dots, \chi_k) = \int_{G^{(k)}} \prod_{i=1}^k \overline{\chi_i(\xi_i)} h(\xi_1, \dots, \xi_k) \mu^{(k)}(d\xi_1 \times \dots \times d\xi_k)$$

where $\chi_i \in \hat{G}$ and $\bar{}$ denotes complex conjugation. For $f \in L_{1,r}(G)$ let $\sigma(f) = \{\chi: \chi \in \hat{G}, \hat{f}(\chi) \neq 0\}$.

If K is a group and H a subset of K , we shall say that a mapping $\phi: H \rightarrow C = \{\exp(2\pi ix): 0 \leq x < 1\}$ acts homomorphically on H to C if

- (i) $\phi(a^{-1}) = [\phi(a)]^{-1}$ whenever $a, a^{-1} \in H$.
- (ii) For each N , $\phi(\prod_{i=1}^N a_i) = \prod_{i=1}^N \phi(a_i)$ whenever a_1, a_2, \dots, a_N and $\prod_{i=1}^N a_i$ are in H .

3. Main theorem.

THEOREM. $\{\rho_k(\cdot): k=1, 2, \dots\}$ is a complete set of translation invariants for $L_{1,r}(G)$.

LEMMA 1. Let H be a symmetric subset of a group K (i.e., if $a \in H$ then $a^{-1} \in H$). If ϕ acts homomorphically on H to C then ϕ can be extended to a homomorphism of the subgroup $[H]$, generated by H , to C .

PROOF. For $c \in [H]$ define $\phi(c) = \phi(a_1)\phi(a_2) \dots \phi(a_p)$ where $c = a_1 a_2 \dots a_p$, $a_i \in H$ ($1 \leq i \leq p$). To see that this definition of $\phi(c)$ is consistent let us suppose $c = b_1 b_2 \dots b_q$ with $b_i \in H$ ($1 \leq i \leq q$). Then $b_q = a_1 a_2 \dots a_p b_1^{-1} b_2^{-1} \dots b_{q-1}^{-1}$ so that $\phi(b_q) = \phi(a_1)\phi(a_2) \dots \phi(a_p)\phi(b_1^{-1})\phi(b_2^{-1}) \dots \phi(b_{q-1}^{-1})$ by (ii) above. By (i) $\phi(b_i^{-1}) = [\phi(b_i)]^{-1}$ so that $\phi(b_1)\phi(b_2) \dots \phi(b_q) = \phi(a_1)\phi(a_2) \dots \phi(a_p)$. The extended mapping is easily seen to be a homomorphism of $[H]$ into C .

LEMMA 2. If K is a topological group, H an open symmetric subset of K and ϕ a continuous mapping $\phi: H \rightarrow C$ which acts homomorphically on H to C , then ϕ can be extended to a continuous homomorphism of the (necessarily closed) subgroup $[H]$, generated by H , to C .

PROOF. By Lemma 1, ϕ can be extended to ϕ^* an algebraic homomorphism of $[H]$ to C . It suffices to verify that ϕ^* is continuous. Let H_n denote the set $H \cdot H \cdot \dots \cdot H$ (n copies);² H_n is clearly an open subset of K . Let ϕ_n denote the restriction of ϕ^* to H_n . Clearly ϕ_n is continuous on H_n and since $[H] = \bigcup_{n=1}^{\infty} H_n$, ϕ^* is continuous on $[H]$. As $[H]$ is an open subgroup of K it follows that it is necessarily a closed subgroup [3, p. 37].

² $H_n = \{c: c = a_1 a_2 \dots a_n, a_i \in H\}$.

PROOF OF THE THEOREM. If $f \in L_{1,r}(G)$ then

$$\hat{f}(\chi^{-1}) = \overline{\hat{f}(\chi)}$$

and hence $\sigma(f)$ is a symmetric subset of \hat{G} . An elementary computation using Fubini's theorem yields

$$(1) \quad \hat{\rho}_k(f)(\chi_1, \chi_2, \dots, \chi_k) = \hat{f}\left(\prod_{i=1}^k \chi_i^{-1}\right) \prod_{i=1}^k \hat{f}(\chi_i)$$

Suppose $f, g \in L_{1,r}(G)$ with

$$(2) \quad \rho_k(f) = \rho_k(g), \quad k = 1, 2, \dots$$

From (2) with $k=1$ we obtain

$$(3) \quad \hat{f}(\chi)\hat{f}(\chi^{-1}) = |\hat{f}(\chi)|^2 = |\hat{g}(\chi)|^2 = \hat{g}(\chi)\hat{g}(\chi^{-1})$$

so that we may write

$$(4) \quad \hat{g}(\chi) = \phi(\chi)f(\chi)$$

where $|\phi(\chi)| = 1$ for $\chi \in \sigma(f)$. The reality of f and g and (4) implies moreover

$$(5) \quad \phi(\chi^{-1}) = \overline{\phi(\chi)} = [\phi(\chi)]^{-1}$$

again for $\chi \in \sigma(f)$. Equations (1), (2), (4) and (5) yield

$$(6) \quad \phi\left(\prod_{i=1}^N \chi_i\right) = \prod_{i=1}^N \phi(\chi_i)$$

whenever

$$\chi_1, \chi_2, \dots, \chi_N, \prod_{i=1}^N \chi_i \in \sigma(f).$$

Since $\hat{f}(\cdot)$ is continuous and does not vanish on $\sigma(f)$, $\phi(\cdot)$ is continuous on $\sigma(f)$ and $\sigma(f)$ is open. Thus ϕ is continuous and acts homomorphically on the open symmetric subset $\sigma(f)$ to C . By Lemma 2, ϕ can be extended to a continuous homomorphism of $[\sigma(f)]$ to C . The mapping $\phi: [\sigma(f)] \rightarrow C$ can be extended [4, p. 138] to a character of \hat{G} . Thus we have

$$(7) \quad \hat{g}(\chi) = \phi(\chi)\hat{f}(\chi)$$

for all $\chi \in \hat{G}$, with ϕ a character of \hat{G} . By the duality theorem [1, p. 151]

$$(8) \quad \phi(\chi) = \chi(a)$$

for some fixed $a \in G$ and therefore the uniqueness theorem [1, p. 146] finally yields

$$(9) \quad g(x) = f(ax) \quad (\mu \text{ a.e.}).$$

4. Counterexamples. We will now exhibit a counterexample to the conjecture

$$\rho_1(f) = \rho_1(g) \Rightarrow g \in \mathfrak{I}(f).$$

Let G be the real line and A a subset of G of finite positive measure which is not a translate of its inverse set. If f and g are the characteristic functions of A and $-A = \{-x: x \in A\}$ respectively then $\rho_1(f)(x) = \mu(A \cap (A-x)) = \mu(-A \cap (-A-x)) = \rho_1(g)(x)$ while $g \notin \mathfrak{I}(f)$. Similarly the conjecture

$$\rho_1(f) = \rho_1(g) \Rightarrow g \in \mathfrak{I}(f) \cap \mathfrak{I}(f^-)$$

where $f^-(x) = f(x^{-1})$ is also false. To show this let A and B be subsets of the real line as described above and f and g the characteristic functions of $A \times B$ and $A \times (-B)$. Then f and g have the same (first order) autocorrelation function while $g \notin \mathfrak{I}(f) \cup \mathfrak{I}(f^-)$.

It is likewise conjectured by the authors that no finite subset of $\{\rho_k(\cdot): k=1, 2, \dots\}$ is a complete set of translation invariants unless restrictions are placed upon the functions. In this connection the following corollary is easily obtained.

COROLLARY. *If \mathfrak{U} , is the subset of $L_{1,r}(G)$ consisting of those f for which $\sigma(f) \cdot \sigma(f) \cdot \dots \cdot \sigma(f)$ (ν copies) is a subgroup of \hat{G} then $\{\rho_k(\cdot): k=1, 2, \dots, 3\nu-1\}$ is a complete set of translation invariants for \mathfrak{U} . In particular if $\hat{f}(\chi) \neq 0$ ($\chi \in \hat{G}$) and $\rho_i(f) = \rho_i(g)$ ($i=1, 2$) then $g \in \mathfrak{I}(f)$.*

5. Further considerations. For complex-valued μ -integrable functions $\sigma(f)$ is no longer symmetric in general and our argument fails. In fact if G is the unit interval and $f \in L_1(G)$ with

$$\int_0^1 f(\xi) \exp(-2\pi i n \xi) d\xi = 0 \begin{cases} n < 0, \\ n > N \end{cases}$$

then $\rho_k(f) \equiv 0$ for all k . Minor modifications of the definition of the autocorrelation function do not substantially change the situation.

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