NOTE ON A THEOREM OF NEHARI
R. M. MORONEY

In a recent paper [1] Nehari investigated the oscillation of solutions $y(x)$ of

$$y'' + yF(y^2, x) = 0$$

under the following conditions on the function $F(t, x)$:

(2a) $F(t, x)$ is continuous in $(t, x)$ on $\{(t, x): 0 \leq t < \infty, 0 < x < \infty\}$.

(2b) $F(t, x) > 0$ for $t > 0, x > 0$.

(2c) For fixed positive $x$ and some $\varepsilon > 0$

$$t_2^{-1}F(t_2, x) > t_1^{-1}F(t_1, x) \quad (0 \leq t_1 < t_2 < \infty).$$

In the course of this investigation the question arose whether a $C^2$ solution of (1), with $F$ subject to (2), is uniquely determined by the conditions

(C) $y(a) = y'(b) = 0$, $y(x) > 0$ for $x \in (a, b]$.

Nehari conjectured that this solution is unique under an additional condition on the behavior of $F(t, x)$ as a function of $x$. The purpose of this note is to show that such is the case if $F(t, x)$ satisfies the following condition:

(2d) For each fixed positive $\rho$ and $0 \leq x_1 < x_2 < \infty$, $F(\rho, x_2) \geq F(\rho, x_1)$.

We formulate this as a theorem:

**Theorem.** In (1), let $F(t, x)$ satisfy hypotheses (2a) to (2d). Then for each pair $(a, b)$, $0 \leq a < b < \infty$, there exists a unique solution $y(x)$ of (1) on $[a, b]$ satisfying (C) and having two continuous derivatives on $[a, b]$.

**Lemma 1.** Let $y_1(x)$ and $y_2(x)$ be two $C^2$ solutions of (1) on some interval $[a, c)$ such that

(a) $y_1(a) = y_2(a) = 0$, $y_1'(a) = y_2'(a) > 0$.

(b) $y_1(x) > 0$ and $y_1'(x) > 0$ for $x$ in $(a, c)$.

Then $y_1(x) = y_2(x)$ on $a \leq x < c$.

**Proof.** Suppose $y_2(x) > y_1(x)$ for some $x$ in $(a, c)$. Then by the mean-value theorem there exists an $x_3$ in $(a, x)$ such that $y_2'(x_3) > y_1'(x_3)$ and by a second application an $x_3$ in $(a, x_2)$ such that $y_2''(x_3) > y_1''(x_3)$. Because of (2b) and the form of (1), however, this implies $y_2(x_3) < y_1(x_3)$. 

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By repeating the foregoing argument one sees that if \( y_2(x) \) and \( y_1(x) \) differ at any point of \((a, c)\) a situation as in Figure 1 must arise, that is there will exist an \( x \) interval \([t_1, t_2] \subset (a, c)\) such that \( y_1(t_i) = y_2(t_i) \), \( y_2(x) > y_1(x) \) on \( t_1 < x < t_2 \), and \( y'_2(t_i) > y'_1(t_i) \), \( y'_2(t_2) < y'_1(t_2) \).

We now show that this is impossible.

By the continuity of \((y_2 - y_1)'\) as a function of \( y \), there will exist \( t_3 \) and \( t_4 \) such that \( y_2(t_3) = y_1(t_4) = \gamma \), \( y'_2(t_3) = y'_1(t_4) = z \), \( t_3 < t_4 \) (see Figure 2).

By (1), however, on \( t_1 \leq x \leq t_2 \)

\[
\begin{align*}
\text{(3)} \quad y_2(x) &= y_2(t_i) + (x - t_i)y'_2(x) + \int_{t_1}^{x} (s - t_1)y_2(s)F(y_2(s), s)ds, \\
\text{(4)} \quad y_1(x) &= y_1(t_i) + (x - t_i)y'_1(x) + \int_{t_1}^{x} (s - t_1)y_1(s)F(y_1(s), s)ds,
\end{align*}
\]

as is easily seen by differentiating. Using (3) at \( t_3 \) and (4) at \( t_4 \) and subtracting gives

\[
0 = (t_3 - t_4)z + \int_{t_1}^{t_4} (s - t_1)y_2(s)F(y_2(s), s)ds - \int_{t_1}^{t_4} (s - t_1)y_1(s)F(y_1(s), s)ds.
\]
In the first integral make the change of variable $\lambda = y_2(s)$, and in the second $\lambda = y_1(s)$:

$$0 = (t_3 - t_4)z + \int_{p}^{r} \frac{\lambda}{\lambda_4(\lambda)} F(\lambda^2, s_2(\lambda)) d\lambda$$

(6)

$$- \int_{p}^{r} (s_1 - t_1)(\lambda) \frac{\lambda}{\gamma'_1(\lambda)} F(\lambda^2, s_1(\lambda)) d\lambda.$$ 

Since for each $\lambda$ in $[p, r] (s_1 - t_1)(\lambda) \geq (s_2 - t_1)(\lambda)$ and $y'_1(\lambda) \leq y'_2(\lambda)$, while by hypothesis (2d) $F(\lambda^2, s_2(\lambda)) \leq F(\lambda^2, s_1(\lambda))$, the difference between the integrals in (6) is negative. So is the first term, however, so the right side can not be zero.

**Lemma 2.** Let $y_2(x)$ and $y_1(x)$ be solutions of (1) on $[a, b]$ such that $(i = 1, 2)$

$$y_i \in C^2[a, b]$$

$$y_i(a) = y_i'(b) = 0$$

$$y_i(x) > 0$$

for $x \in (a, b)$.

Then $y'_2(a) = y'_1(a)$.

**Proof.** Suppose $y'_2(a) > y'_1(a)$, and consider the function $(y_2y'_1 - y_1y'_2)(x)$. We have

$$(y_2y'_1 - y_1y'_2)' = y_2y'_1[F(y_2, x) - F(y_1, x)]$$
so

\[(7) \quad (y_2 y'_1 - y_1 y'_2)(x) = \int_a^x y_2(s)y_1(s)[F(y_2(s), s) - F(y_1(s), s)]ds.\]

By hypothesis \((y_2 y'_1 - y_1 y'_2)(b) = 0\). The integrand in (7) is positive, however, as long as \(y_2(s) > y_1(s)\) and in particular on some interval \((a, b)\)—because \((y_2 - y_1)'(a) > 0\). In fact, \(a\) may be taken as \(b\) because the same argument as in Lemma 1 shows that the graphs of \(y_2(x)\) and \(y_1(x)\) can not intersect on \((a, b)\). Thus the right side of (7) does not tend to zero as \(x \to b^+\).

**Proof of the theorem.** The existence of at least one solution of \((1) + (C)\) has been proved by Nehari [1, Theorem IV]. By the preceding lemmas there is at most one such solution.

**Reference**


**Massachusetts Institute of Technology**

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**ON THE MEASURABILITY OF FUNCTIONS IN TWO VARIABLES**

MARK MAHOWALD

Let \((X, \mu)\) and \((Y, \nu)\) be two compact spaces having regular Borel measures defined on them. By a measurable modification \(f(x, y)\) of a function \(f(x, y)\) we mean a function measurable in both variables together and for which \(f(x, -) = f(x, \cdot)\) almost everywhere \([\nu]\) for every \(x\). The purpose of this note is to prove the following theorem.

**Theorem.** If \(Y\) is metric and if \(f(x, y)\) has a measurable modification and \(f(x, \cdot)\) is continuous for almost all \(x\), then \(f(x, y)\) is measurable in both variables together.

This theorem was discovered in an effort to prove that the Nelson canonical version [2] is measurable if it has a measurable modification. The theorem would prove this result except for the restriction that \(Y\) be metric.

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