NOTE ON A THEOREM OF NEHARI

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In a recent paper [1] Nehari investigated the oscillation of solutions $y(x)$ of

$$y'' + yF(y^2, x) = 0$$

under the following conditions on the function $F(t, x)$:

(2a) $F(t, x)$ is continuous in $(t, x)$ on $\{(t, x) : 0 \leq t < \infty, 0 < x < \infty\}$.

(2b) $F(t, x) > 0$ for $t > 0$, $x > 0$.

(2c) For fixed positive $x$ and some $\epsilon > 0$

$$t_2^{-1}F(t_2, x) > t_1^{-1}F(t_1, x) \quad (0 \leq t_1 < t_2 < \infty).$$

In the course of this investigation the question arose whether a $C^2$ solution of (1), with $F$ subject to (2), is uniquely determined by the conditions

(C) $y(a) = y'(a) = 0$, $y(x) > 0$ for $x \in (a, b]$.

Nehari conjectured that this solution is unique under an additional condition on the behavior of $F(t, x)$ as a function of $x$. The purpose of this note is to show that such is the case if $F(t, x)$ satisfies the following condition:

(2d) For each fixed positive $\rho$ and $0 \leq x_1 < x_2 < \infty$, $F(\rho, x_2) \geq F(\rho, x_1)$.

We formulate this as a theorem:

**Theorem.** In (1), let $F(t, x)$ satisfy hypotheses (2a) to (2d). Then for each pair $(a, b)$, $0 \leq a < b < \infty$, there exists a unique solution $y(x)$ of (1) on $[a, b]$ satisfying (C) and having two continuous derivatives on $[a, b]$.

**Lemma 1.** Let $y_1(x)$ and $y_2(x)$ be two $C^2$ solutions of (1) on some interval $[a, c)$ such that

(a) $y_1(a) = y_2(a) = 0$, $y'_1(a) = y'_2(a) > 0$.

(b) $y'_1(x) > 0$ and $y''_1(x) > 0$ for $x \in (a, c)$.

Then $y_1(x) = y_2(x)$ on $a \leq x < c$.

**Proof.** Suppose $y_2(x) > y_1(x)$ for some $\hat{x}$ in $(a, c)$. Then by the mean-value theorem there exists $x_2$ in $(a, \hat{x})$ such that $y_2'(x_2) > y_1'(x_2)$ and by a second application an $x_3$ in $(a, x_2)$ such that $y_2''(x_3) > y_1''(x_3)$. Because of (2b) and the form of (1), however, this implies $y_2(x_3) < y_1(x_3)$.

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By repeating the foregoing argument one sees that if \( y_2(x) \) and \( y_1(x) \) differ at any point of \((a, c)\) a situation as in Figure 1 must arise, that is there will exist an \( x \) interval \([t_1, t_2] \subseteq (a, c)\) such that \( y_1(t_i) = y_2(t_i) \), \( y_2(x) > y_1(x) \) on \( t_1 < x < t_2 \), and \( y_2'(t_i) > y_1'(t_i) \), \( y_2'(t_2) < y_1'(t_2) \). We now show that this is impossible.

By the continuity of \((y_2 - y_1)'\) as a function of \( y \), there will exist \( t_3 \) and \( t_4 \) such that \( y_2(t_3) = y_1(t_4) = r \), \( y_2'(t_4) = y_1'(t_4) = z \), \( t_3 < t_4 \) (see Figure 2).

By (1), however, on \( t_1 \leq x \leq t_2 \)

\[
\begin{align*}
(3) \quad y_2(x) &= y_2(t_i) + (x - t_i) y_2'(x) + \int_{t_1}^{x} (s - t_i) y_2(s) F(y_2(s), s) \, ds, \\
(4) \quad y_1(x) &= y_1(t_i) + (x - t_i) y_1'(x) + \int_{t_1}^{x} (s - t_i) y_1(s) F(y_1(s), s) \, ds,
\end{align*}
\]

as is easily seen by differentiating. Using (3) at \( t_3 \) and (4) at \( t_4 \) and subtracting gives

\[
\begin{align*}
0 &= (t_3 - t_4) z + \int_{t_1}^{t_3} (s - t_1) y_2(s) F(y_2(s), s) \, ds \\
&\quad - \int_{t_1}^{t_4} (s - t_1) y_1(s) F(y_1(s), s) \, ds.
\end{align*}
\]
In the first integral make the change of variable $\lambda = y_2(x)$, and in the second $\lambda = y_1(x)$:

$$0 = (t_3 - t_4)z + \int_p^r (s_2 - t_3)(\lambda) \lambda \lambda \lambda \lambda \lambda F(\lambda^2, s_2(\lambda))d\lambda$$

$$- \int_p^r (s_1 - t_1)(\lambda) \lambda \lambda \lambda \lambda \lambda F(\lambda^2, s_1(\lambda))d\lambda.$$

Since for each $\lambda$ in $[p, r]$ $(s_1 - t_1)(\lambda) \geq (s_2 - t_1)(\lambda)$ and $y_1'(\lambda) \leq y_2'(\lambda)$, while by hypothesis (2d) $F(\lambda^2, s_1(\lambda)) \leq F(\lambda^2, s_1(\lambda))$, the difference between the integrals in (6) is negative. So is the first term, however, so the right side can not be zero.

**Lemma 2.** Let $y_2(x)$ and $y_1(x)$ be solutions of (1) on $[a, b]$ such that

$$(i=1, 2)$$

$$y_i \in C^2[a, b]$$

$$y_i(a) = y_i'(b) = 0$$

$$y_i(x) > 0 \quad \text{for } x \in (a, b].$$

Then $y_2'(a) = y_1'(a)$.

**Proof.** Suppose $y_2'(a) > y_1'(a)$, and consider the function $(y_2y_1' - y_1y_2')(x)$. We have

$$(y_2y_1' - y_1y_2')(x) = y_2y_1[F(y_2, x) - F(y_1, x)]$$
so

(7) \((y_2y_2' - y_1y_1')(x) = \int_a^x y_2(s)y_1(s)[F(y_2(s), s) - F(y_1(s), s)]ds.\)

By hypothesis \((y_2y_2' - y_1y_1')(b) = 0\). The integrand in (7) is positive, however, as long as \(y_2(s) > y_1(s)\) and in particular on some interval \((a, b)\)—because \((y_2 - y_1)'(a) > 0\). In fact, \(\alpha\) may be taken as \(b\) because the same argument as in Lemma 1 shows that the graphs of \(y_2(x)\) and \(y_1(x)\) can not intersect on \((a, b)\). Thus the right side of (7) does not tend to zero as \(x \to b^+\).

**Proof of the theorem.** The existence of at least one solution of (1)+(C) has been proved by Nehari [1, Theorem IV]. By the preceding lemmas there is at most one such solution.

**Reference**


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**ON THE MEASURABILITY OF FUNCTIONS IN TWO VARIABLES**

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Let \((X, \mu)\) and \((Y, \nu)\) be two compact spaces having regular Borel measures defined on them. By a measurable modification \(f(x, y)\) of a function \(f(x, y)\) we mean a function measurable in both variables together and for which \(f(x-) = f(x-) \) almost everywhere \([\nu]\) for every \(x\).

The purpose of this note is to prove the following theorem.

**Theorem.** If \(Y\) is metric and if \(f(x, y)\) has a measurable modification and \(f(x-)\) is continuous for almost all \(x\), then \(f(x, y)\) is measurable in both variables together.

This theorem was discovered in an effort to prove that the Nelson canonical version [2] is measurable if it has a measurable modification. The theorem would prove this result except for the restriction that \(Y\) be metric.

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