AXIOMS THAT DEFINE SEMI-METRIC, MOORE, AND METRIC SPACES

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In [1] L. F. McAuley asked the following question: is it possible to partition Moore’s metrization theorem into three or more parts which begins with a condition for a topological space and which ends with a condition for a metrizable space, but with necessary and sufficient conditions somewhere between these extremes for semi-metric and Moore spaces? Axiom Z, stated below, is such a partitioning. The notation “Axiom Z_i” denotes parts (1), (2), • • • , (i) of Axiom Z. In §1 it is proved in Theorems 1, 2, and 3, respectively, that a necessary and sufficient condition for a topological space to be semi-metrizable, a Moore space, and metrizable is that it satisfy Axiom Z_2, Axiom Z_3, and Axiom Z_4 respectively. A counter-example is given in §2 which shows that the argument for the statement a Moore space is a semi-metric topological space in Theorem 6.2 in [1] is not correct. Finally, in §3 it is shown that part (3) of Theorem 2 in [2] can be changed so that the resulting statement is equivalent to a Moore space. Definitions are given in [1].

Definition. If \{J_n\} denotes a sequence such that for each natural number n, J_n denotes a collection of neighborhoods covering a point set M, then the sequence \{B_i\}, where i denotes a natural number, is said to be a basic refinement of \{J_n\} for M provided that with each point p in M there is associated a sequence \{b_i(p)\} such that for each i: (1) b_i(p) is a neighborhood in \{J_n\}, (2) b_{i+1}(p) is a subset of b_i(p), (3) p is the only point common to \{b_i(p)\}, and (4) B_i denotes the collection of all neighborhoods b_i(p) for all points in M.

1. Axiom Z.

Axiom Z. Let T denote a topological space in which there exists a sequence \{J_n\} such that:

(1) for each natural number n, J_n denotes a collection of neighborhoods in T covering T,

(2) there exists a basic refinement \{B_n\} of \{J_n\} for T such that if M denotes a point set and p denotes a point, then either (a) every neighborhood containing p contains a point in (M — p) or (b) there exists an n such that if x denotes a point in (M — p), b_n(p) does not contain x and b_n(x) does not contain p,

(3) if R denotes a neighborhood containing p and x is in R, then there

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exists an $n$ such that if $g$ denotes a neighborhood in $J_n$ that contains $p$, then $g$ is a subset of $R$ not containing $x$ unless $x$ is $p$, and

(4) if $R$ contains $p$ and $x$ is different from $p$, then there exists an $n$ such that if each of $h$ and $k$ denotes a neighborhood in $J_n$, $k$ contains $p$, and $h$ and $k$ have a common part, then $h$ is a subset of $R$ not containing $x$.

Parts (1) and (4) are Moore’s metrization theorem and parts (1) and (3) are parts (1) and (3) of Moore’s Axiom 13.

**Theorem 1.** A necessary and sufficient condition for a topological space $T$ to be semi-metrizable is that $T$ satisfy Axiom Z2.

**Proof of necessity.** It follows from the definition of a semi-metric topological space that for each point $p$ in $T$, there exists a sequence $\{b_n(p)\}$ such that for each $n$, (1) $b_n(p)$ denotes a neighborhood containing $p$, (2) $b_n(p)$ is a subset of the $1/n$-neighborhood of $p$, and (3) $b_{n+1}(p)$ is a subset of $b_n(p)$. Now, for each $n$, let $B_n$ denote the collection of all the neighborhoods $b_n(p)$ for the various points in $T$. Finally, for natural numbers $i$ and $m$ let $J_n$ denote the collection of all the elements in $B_i$ for all $i \geq m$. It is not difficult to verify that Axiom Z2 is satisfied by the sequences $\{J_n\}$ and $\{B_n\}$.

**Proof of sufficiency.** For each pair of points $p$ and $x$ in $T$, let $n$ denote the smallest $i$ such that $b_i(x)$ does not contain $p$ and $b_i(p)$ does not contain $x$ where $b_i(x)$ and $b_i(p)$ belong to $B_i$. Now let $d(p, x) = d(x, p) = 1/n$. Define $d(p, p) = 0$. It follows that $d$ is a semi-metric for $T$ and that the sufficiency of the condition is established.

**Theorem 2.** A necessary and sufficient condition for a topological space $T$ to be a Moore space is that $T$ satisfy Axiom Z3.

**Proof of necessity.** Let $\{G_n\}$ denote the sequence of collections of neighborhoods given in Moore’s Axiom 13. Since a Moore space is a semi-metric space [1], define the sequences $\{B_n\}$ and $\{J_n\}$ in a manner analogous to that in the proof of the necessity of Theorem 1 with the additional restriction that each member of $J_n$ be a subset of an element of $G_n$, i.e., $J_n$ refines $G_n$ for each $n$. It follows that the sequences $\{J_n\}$ and $\{B_n\}$ satisfy Axiom Z3.

**Proof of sufficiency.** Given that $T$ satisfies Axiom Z3, let $G_n$ denote the collection of all elements $B_i$ for all $i \geq n$. It follows that the sequence $\{G_n\}$ satisfies Moore’s Axiom 13.

**Theorem 3.** A necessary and sufficient condition for a topological space $T$ to be metrizable is that $T$ satisfy Axiom Z4.

2. A correction. The following counter-example shows that the argument for the statement a Moore space is a semi-metric topological space in [1, Theorem 6.2] is not correct.

Let the points of space be the points of the plane on or above the 
$x$-axis. Neighborhoods are of two types—interiors of circles above the 
$x$-axis and interiors of circles tangent to the $x$-axis from above to-
gether with the point of tangency. Let $G_i$ denote the collection of all 
these neighborhoods whose diameters are less than $1/i$.

McAuley defined the distance between two points $p$ and $q$ as fol-
lows: denote by $n$ the least positive integer such that if $g(p)$ and $g(q)$ de-
note two neighborhoods in $G_i$ containing the points $p$ and $q$ respec-
tively, then $g(p) \cdot g(q) = 0$. Define $d(p, q) = 1/n$. By this definition, a point on 
the $x$-axis is a distance limit point of the $x$-axis but is not a limit point
of the $x$-axis.

It is not difficult to show that the following definition of distance
is sufficient to show that a Moore space is a semi-metric topological
space. Let $n$ denote the least natural number $i$ such that if $g(p)$ and $g(q)$
are two neighborhoods in $G_i$ containing $p$ and $q$ respectively, then $g(p)$
does not contain $q$ and $g(q)$ does not contain $p$. Define $d(p, q) = 1/n$.

3. **Axiom A.** R. L. Moore’s Axiom 13 is stated in terms of a sequence 
of collections of regions covering space. The following axiom is stated
in terms of a point and a sequence of regions containing the point.
The axiom is the same as Theorem 2 in [2] except for part (3).

**Axiom A.** If $p$ denotes a point, there exists a sequence \{${R_n(p)}$\} where
for each natural number $n$, $R_n(p)$ denotes a region containing $p$ such
that:

1. $p$ is the only point common to \{${R_n(p)}$\},
2. for each $n$, $R_{n+1}(p)$ is a subset of $R_n(p)$,
3. if $R$ denotes a region containing $p$, then there exists an $n$ such that
if $z$ denotes any point and $R_n(z)$ contains $p$, $R_n(z)$ is a subset of $R$.

**Theorem 4.** In the presence of Moore’s Axiom 0, Axiom 13 is equiv-
lent to Axiom A.

**Proof.** That Axiom 13 implies Axiom A is established by obtaining
the sequence \{${R_n(p)}$\} as in the proof of Theorem 2 in [2] and show-
ing that part (3) of Axiom A follows from part (3) of Axiom 13. To
show that Axiom A implies Axiom 13, let $G_n$, for each $n$, denote the
collection of all regions $R_i(p)$ for all natural numbers $i \geq n$ for all
points $p$. It is easy to verify that the sequence \{${G_n}$\} satisfies Axiom 13.

**References**

1. **L. F. McAuley**, *A relation between perfect separability, completeness, and normal-

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