UPPER AND LOWER BOUNDS OF THE NORM OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Consider the differential system

\[ z' = f(x, z) \]

under the assumptions:

(i) \( x \) is a real variable, \( z \) and \( f \) are finite dimensional complex vectors with \( n \) components \( z_i \) and \( f_i \) respectively,

(ii) \( f \) is continuous in \((x, z)\) for all \( z \) and for all \( x \) in \( a \leq x \leq b \). Define

\[ |z| = \sum_{i=1}^{n} |z_i|. \]

Then we have the following

**Theorem 1.** Let the function \( g(x, u) \geq 0 \) be continuous in the region \( a \leq x \leq b, \ u \geq 0 \). Let the function \( f(x, z) \) of (1) satisfy the condition

\[ |f(x, z)| \leq g(x, |z|). \]

Let \( z(x) \) satisfy \( |z(x)| > 0 \) and be a solution of (1) in the region \( a \leq x \leq b \). Then for all \( x \) in \( a \leq x \leq b \), we have

\[ |z(x)| \leq M(x) \]

and

\[ |z(x)| \geq m(x) \]

where \( M(x) \) and \( m(x) \) are the maximal and minimal solutions of \( u' = \pm g(x, u) \), \( u(a) = |z(a)| \), respectively.

**Proof.** The inequality (2) follows from the Theorem 1 in [4]. To prove (3), we have to use essentially the same argument as in [4] but now we have to consider the minimal solution of \( u' = -g(x, u) \), \( u(a) = |z(a)| \) instead of the maximal solution of \( u' = g(x, u) \), \( u(a) = |z(a)| \). This completes the proof.

**Remark.** The above theorem includes the results of Bellman [1], Bihari [2] and Langenhop [3] as special cases. Taking \( g(x, u) = v(x)u \), it is easy to see that \( M(x) = u(a)\exp\left[\int_{a}^{x} v(s)ds\right] \) and \( m(x) = u(a)\exp\left[-\int_{a}^{x} v(s)ds\right] \), which correspond to Bellman’s results. Suppose \( g(x, u) = v(x)g(u) \), where \( g(u) > 0 \) for \( u > 0 \). Then we can easily obtain \( M(x) = G^{-1}[G(u(a)) + \int_{a}^{x} v(s)ds] \) and

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\[ m(x) = G^{-1}\left[G(u(a)) - \int_a^x v(s)ds\right], \]

where \( G(u) = \int_{u_0}^{u} [g(r)]^{-1}dr, \ u_0 \geq 0, \) which are exactly the results of Langenhop. This implies that the monotonicity assumption regarding \( g(u) \) in his hypotheses is superfluous.

We can formulate an interesting comparison theorem. Consider another differential system

\[(4) \quad z' = h(x, z)\]

under the assumptions:

(i) \( x \) is a real variable, \( z \) and \( h \) are finite dimensional complex vectors with \( n \) components \( z_i \) and \( h_i \) respectively,

(ii) \( h \) is continuous in \( (x, z) \) for all \( z \) and for all \( x \) in \( a \leq x \leq b. \)

**Theorem 2.** Let the function \( g(x, u) \geq 0 \) be continuous in a region \( a \leq x \leq b, \ u \geq 0. \) Let the functions \( f(x, z) \) and \( h(x, z) \) satisfy the condition

\[ |f(x, z_1) - h(x, z_2)| \leq g(x, |z_1 - z_2|). \]

Let the functions \( z_1(x) \) and \( z_2(x) \) be solutions of (1) and (4) respectively and satisfy \( |z_1(x) - z_2(x)| > 0. \) Then for all \( x \) in \( a \leq x \leq b, \) we have

\[ |z_1(x) - z_2(x)| \leq M(x); \quad |z_1(x) - z_2(x)| \geq m(x), \]

where \( M(x) \) and \( m(x) \) are the maximal and minimal solutions of \( u' = \pm g(x, u); \ u(a) = |z_1(a) - z_2(a)|, \) respectively.

The proof is similar to that of the Theorem 1 and hence omitted.

(Professor R. P. Boas informed me that essentially the same results by the same methods have been obtained independently by A. D. Zeibur.)

**References**


