SELF-ADJOINT TOEPLITZ OPERATORS AND ASSOCIATED ORTHONORMAL FUNCTIONS

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1. Introduction. Let $l^2$ be the Hilbert space of square summable sequences $f = (f_0, f_1, f_2, \ldots)$. $l^2$ is isomorphic to the space $H^2$ of functions holomorphic in the unit disk $\Delta$ with square integrable boundary values, under the map

\begin{equation}
 f \rightarrow F, \quad F(z) = \sum_{n=0}^{\infty} f_n z^n.
\end{equation}

A Toeplitz operator is an $l^2$ linear operator $T$ to which corresponds a function $W$ on the unit circle $\Gamma$, such that under the isomorphism (1.1) we have for the inner product

\begin{equation}
 \langle Tf, g \rangle = \frac{1}{2\pi} \int_{\Gamma} W(\theta) F(e^{i\theta}) G^*(e^{i\theta}) d\theta.
\end{equation}

Here $*$ denotes complex conjugation.

This work is concerned with the concrete spectral theory of Toeplitz operators that are associated with functions $W$ that satisfy the following two hypotheses:

(i) $W$ is real, bounded below, and absolutely integrable on $\Gamma$, but it is not equivalent to a constant function.

(ii) For each real $\lambda$ the set $\Gamma_\lambda = \{ \theta : W(\theta) \leq \lambda \}$ is, modulo a set of measure zero, an arc of the circle.

By concrete spectral theory we mean that we exhibit an explicit sigma-finite measure $\rho$ on $(-\infty, \infty)$ and an explicit unitary correspondence $U: \ell^2 \rightarrow L^2(\rho)$ such that $UTU^{-1} = M$, where $M$ is the multiplication operator on $L^2(\rho)$ which sends $g(\lambda)$ into $\lambda g(\lambda)$. Hypothesis (i) implies that $T$ is bounded below, so its Friedrichs extension (again named $T$) is self-adjoint.

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The results obtained are conveniently described in terms of the set of vectors $k(u) \in \ell^2$ defined for each $u \in \Delta$ by

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2 This description applies only to operators with simple spectrum. The hypotheses given guarantee that $T$ has simple spectrum, but this is not in general true for Toeplitz operators, cf. [6].

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Under the correspondence (1.1) it follows that \( K(u; z) = (1 - uz)^{-1} \), and thus for each \( f \in L^2 \) we have

\[
(1.4) \quad \langle f, k(u^*) \rangle = F(u).
\]

Let \( E \) be the spectral measure of \( T \). In Theorem 3 we find that there is an absolutely continuous measure \( \rho \) on the line given by (3.3) and a collection of functions \( \Phi(u; \lambda) \) given by (3.2) such that for all \( u, v \in \Delta \) and each real Borel set \( \Lambda \)

\[
(1.5) \quad \langle E(\Lambda)k(u), k(v) \rangle = \int_{\Lambda} \Phi(u; \lambda)\Phi^*(v, \lambda)d\rho(\lambda).
\]

According to (2.7) and Lemma 1 the functions \( \Phi(u; \lambda) \) are for almost all \( \lambda \) holomorphic functions of \( u \in \Delta \) with the Maclaurin expansion

\[
(1.6) \quad \Phi(u; \lambda) = \sum_{n=0}^{\infty} \phi_n(\lambda)u^n.
\]

For fixed \( u \in \Delta, \Phi(u; \lambda) \in L^2(d\rho) \). Now the transformation \( k(u) \rightarrow \Phi(u; \lambda) \) defined on the set \( \mathcal{K} = \{ k(u): u \in \Delta \} \) in \( L^2 \) with range in \( L^2(d\rho) \) preserves inner products as can be seen by taking \( \Lambda = (-\infty, \infty) \) in (1.5). Since \( \mathcal{K} \) is total in \( L^2 \), cf. (1.4), it follows that there exists a unique isometry \( U: L^2 \rightarrow L^2(d\rho) \) such that \( Uk(u) = \Phi(u; \lambda) \). This transformation has the explicit form below obtained from (1.3) and (1.6):

\[
(1.7) \quad Uf = \sum_{n=0}^{\infty} f_n\phi_n.
\]

We next note that (1.5) implies that for each real Borel set \( \Lambda \), \( U \) sends \( E(\Lambda)k(0) \) into the product of the indicator function of \( \Lambda \) and \( \Phi(0; \lambda) \). Since by (3.2) \( \Phi(0; \lambda) \) is almost everywhere nonzero, it follows that the range of \( U \) is \( L^2(d\rho) \). Thus \( U \) is a unitary mapping of \( L^2 \) onto \( L^2(d\rho) \).

As a corollary of the fact that (1.7) is a unitary equivalence we conclude that \( \{ \phi_n \} \) is a complete orthonormal set in \( L^2(d\rho) \). So we see that \( \Phi(u; \lambda) \) is a generating function for a complete orthonormal set of functions in \( L^2(d\rho) \). In the examples at the end we specify \( W \) so as to obtain certain Gegenbauer and Pollaczek polynomials.

2. Analysis of Toeplitz matrices. Suppose now that hypothesis (i) is satisfied, and set \( \lambda_0 = \text{ess inf} \ W \). It is known, [1], that whenever \( \lambda < \lambda_0 \) there is a factorization

\[
(2.1) \quad W(\theta) - \lambda = \left| H_\lambda(e^{i\theta}) \right|^2,
\]
where \( H_\lambda \) is an outer function in \( H^2 \), i.e., the set \( \{ e^{i\lambda}H_\lambda(z) \}_{\lambda=0}^\infty \) is total in \( H^2 \). \( H_\lambda \) is uniquely specified if we impose the normalization \( H_\lambda(0) > 0 \). The explicit formula is

\[
(2.2) \quad H_\lambda(u) = \exp \int_{-\pi}^{\pi} \log (W(\theta) - \lambda)P^*(u^*, \theta) d\theta, \quad u \in \Delta
\]

where

\[
P(u, \theta) = \frac{1}{4\pi} (1 + u e^{i\theta})(1 - u e^{-i\theta})^{-1}.
\]

Thus whenever \( \lambda < \lambda_0 \) and \( T \) is given by (1.2) we may write \( T - \lambda = S^*_h S_\lambda \), where \( S_\lambda \) is the operator which under the map (1.1) transforms into multiplication by \( H_\lambda \), i.e.

\[
(2.3) \quad S_\lambda f \to H_\lambda F.
\]

From (1.4) we easily compute that

\[
(2.4) \quad S^*_h k(u) = H^*_\lambda(u^*) k(u).
\]

Since \( H_\lambda \) is an outer function, \( S_\lambda \) and \( S^*_h \) have densely defined inverses. Hence \( (T - \lambda)^{-1} = S_\lambda^{-1} S^*_h^{-1} \). Using (2.4) we see that

\[
(2.5) \quad \langle (T - \lambda)^{-1} k(u), k(v) \rangle = H^*_\lambda^{-1}(u^*) H^*_\lambda^{-1}(v^*) \langle k(u), k(v) \rangle.
\]

It is useful to rewrite this formula using (2.2) and the fact that \( \langle k(u), k(v) \rangle = (1 - uv)^{-1} \). We obtain

\[
(2.5') \quad \langle (T - \lambda)^{-1} k(u), k(v) \rangle = (1 - uv)^{-1} \exp - \int_{-\pi}^{\pi} \log (W(\theta) - \lambda) \left[ P(u, \theta) + P^*(v, \theta) \right] d\theta.
\]

For \( u, v \in \Delta \), the right hand side of (2.5') is holomorphic in the \( \lambda \)-plane cut along the real axis from \( \lambda_0 \) to \( \infty \). Thus (2.5') provides an analytic continuation of the resolvent of \( T \). We will apply the Stieltjes inversion formula

\[
(2.6) \quad d(E(\lambda) k(u), k(v))/d\lambda = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left\{ \langle (T - \lambda - i\epsilon)^{-1} k(u), k(v) \rangle - \langle (T - \lambda + i\epsilon)^{-1} k(u), k(v) \rangle \right\}
\]

to (2.5'), making use of the following

**Lemma 1.** For almost all real \( \lambda \)

\[
\int_{-\pi}^{\pi} \log |W(\theta) - \lambda| d\theta > - \infty.
\]
Proof. Let \( z = \lambda + i \epsilon, \epsilon > 0 \), and consider \( I(z) = \int_{-\pi}^{\pi} \log |W(\theta) - z| d\theta \).

This equals

\[
 k \int_{-\pi}^{\pi} \log \left[ (W(\theta) - z)(1 + W(\theta)^2)^{-1/2} \right] d\theta,
\]

where \( k = (1/2) \int_{-\pi}^{\pi} \log \left[ 1 + W(\theta)^2 \right] d\theta \).

Thus

\[
 I(z) = \int_{-\infty}^{\infty} \log \left[ (t - z)(1 + t^2)^{-1/2} \right] a(t) dt.
\]

Upon partial integration we have

\[
 I(z) = \int_{\gamma} \left[ (t - z)^{-1} - t(1 + t^2)^{-1} \right] a(t) dt.
\]

Thus \( \lim_{\epsilon \to 0} I(z) \) exists a.e. and \( \lim_{\epsilon \to 0} \Re I(z) = \lim_{\epsilon \to 0} \int_{-\pi}^{\pi} \log |W(\theta) - \lambda + i\epsilon| d\theta \) is finite a.e. The lemma now follows from this by monotone convergence.

The lemma allows us to conclude that

\[
 (2.7) \quad \Psi(u; \lambda) = \exp \left( -\int_{-\pi}^{\pi} \log |W(\theta) - \lambda| P(u, \theta) d\theta \right)
\]

defines for almost all \( \lambda \) a holomorphic function of \( u \in \Delta \). Let \( \Gamma_\lambda = \{ \theta : W(\theta) \leq \lambda \} \). Evidently

\[
 \lim_{\epsilon \to 0} \log (W(\theta) - \lambda \pm i\epsilon) = \log |W(\theta) - \lambda| \quad \text{when} \quad \theta \in \Gamma_\lambda
\]

\[
 = \log |W(\theta) - \lambda| \pm \pi i \quad \text{when} \quad \theta \in \Gamma_\lambda.
\]

We can now calculate the right-hand side of (2.6) from (2.5'). The result is

**Theorem 1.** Let \( T \) be a Toeplitz operator defined by (1.2), where \( W \) satisfies hypothesis (i). Then the resolution of the identity \( E(\lambda) \) of \( T \) satisfies

\[
 d\langle E(\lambda)k(u), k(v) \rangle/d\lambda
\]

\[
 = \pi^{-1} \Psi(u; \lambda) \Psi^*(v; \lambda)(1 - uv^*)^{-1} \sin \left\{ \pi \int_{\Gamma_\lambda} \left[ P(u, \theta) + P^*(v, \theta) \right] d\theta \right\}.
\]

Theorem 1 describes the absolutely continuous part of \( T \). That this is a complete description follows from

**Theorem 2.** Let \( T \) be a Toeplitz operator defined by (1.2) where \( W \) satisfies hypothesis (i). Then the spectral resolution of \( T \) is weakly absolutely continuous with respect to Lebesgue measure.

Proof. See [6]. Theorem 2 makes use of the assumption that \( W \) is not equivalent to a constant. The two theorems combined show that the spectrum of \( T \) is purely continuous and consists of the closed
interval \([\text{ess inf } W, \text{ess sup } W]\). This result is known, cf. [3 and 4]. What remains to be done is to exploit Theorem 1. This will require the use of hypothesis (ii).

3. Concrete spectral theory. We shall need the simple computational

Lemma 2. If \(0 \leq b - a \leq 2\pi\),

\[
\int_a^b P(u, \theta) d\theta = \frac{1}{4\pi} (b - a) + \frac{1}{2\pi i} \log[(1 - u e^{i\alpha})(1 - u e^{i\beta})^{-1}].
\]

Hypothesis (i) says that \(\Gamma_\lambda = \{a(\lambda) \leq \theta \leq b(\lambda)\}\) where \(0 \leq b(\lambda) - a(\lambda) \leq 2\pi\). From Lemma 2 we have

\[
\pi \int_{\Gamma_\lambda} [P(u, \theta) + P^*(v, \theta)] d\theta
\]

\[(3.1)\]

\[
= \frac{1}{2} (b - a) + (2i)^{-1} \log [(1 - u e^{i\alpha})(1 - v e^{-i\beta})(1 - u e^{i\beta})^{-1}(1 - v e^{-i\alpha})^{-1}]
\]

where \(a = a(\lambda)\) and \(b = b(\lambda)\). The main theorem is

Theorem 3. Let \(T\) be a Toeplitz operator defined by (1.2) where \(W\) satisfies hypotheses (i) and (ii). Then in the spectral decomposition of \(T\) we have for each real Borel set \(\Lambda\)

\[(1.5)\]

\[
\langle E(\lambda) k(u), k(v) \rangle = \int_{\Lambda} \Phi(u; \lambda) \Phi^*(v; \lambda) d\rho(\lambda)
\]

where

\[(3.2)\]

\[
\Phi(u; \lambda) = \Psi(u; \lambda)(1 - u e^{i\alpha(\lambda)})^{-1/2}(1 - u e^{i\beta(\lambda)})^{-1/2}
\]

and

\[(3.3)\]

\[
d\rho(\lambda) = \pi^{-1} \sin \frac{1}{2} [b(\lambda) - a(\lambda)] d\lambda.
\]

Proof. With the aid of (3.1) the sine term in the formula of Theorem 1 can be calculated. We obtain

\[
d\langle E(\lambda) k(u), k(v) \rangle / d\lambda = \Phi(u; \lambda) \Phi^*(v, \lambda) \rho'(\lambda)
\]

for almost all \(\lambda\) where \(\rho'(\lambda) = d\rho(\lambda) / d\lambda\). Theorem 2 asserts that the set function \(\langle E(\cdot) k(u), k(v) \rangle\) is absolutely continuous, whence the assertion of Theorem 3 follows.

Corollary 1. The functions \(\Phi(u; \lambda)\) are for almost all \(\lambda\) holomorphic functions of \(u \in \Delta\) with the Maclaurin expansion (1.6). The mapping
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$U: \{f_n\} \to \sum_{n=0}^{\infty} f_n\phi_n$ is a unitary transformation of $l^2$ onto $L^2(d\rho)$ such that $UTU^{-1} = M$, where $M: \gamma(\lambda) \to \lambda g(\lambda)$.

Proof. See the introduction.

4. Examples. The results obtained are:

Example 1. $W(\theta) = \cos \theta$.

\[
\begin{align*}
\rho'(\lambda) &= \pi^{-1}(1 - \lambda^2)^{1/2}, \quad |\lambda| < 1 \\
\Phi(u; \lambda) &= 2^{1/2}(1 - 2\lambda u + u^2)^{-1}, \\
\phi_n(\lambda) &= 2^{1/2}C_n^{(1)}(\lambda), \text{ where } C_n^{(1)}(\lambda)
\end{align*}
\]

is the $n$th Gegenbauer polynomial of order 1, cf. [2, p. 174].

Example 2. $W(\theta) = \sin \theta$. $\rho'(\lambda)$ is as in Example 1.

\[
\begin{align*}
\Phi(u; \lambda) &= 2^{1/2}(1 - 2i\lambda u - u^2)^{-1}, \\
\phi_n(\lambda) &= 2^{1/2}(-i)^nC_n^{(1)}(\lambda).
\end{align*}
\]

Example 3. $W(\theta) = 1$ if $|\theta| < c$; $= 0$ otherwise. Here $0 < c < \pi$.

\[
\rho'(\lambda) = \frac{1}{\pi} \sin c \text{ if } 0 < \lambda < 1, = 0 \text{ otherwise.}
\]

Set $\beta = -1/2\pi \log(\lambda^{-1} - 1)$. Then

\[
\Phi(u; \lambda) = e^{i\theta}(1 + e^{-2i\theta})^{1/2}(1 - ue^{-i\theta})^{-1/2} - i(1 - ue^{i\theta})^{-1/2} + i, \\
\phi_n(\lambda) = e^{i\theta}(1 + e^{-2i\theta})^{1/2}P_n^{(1/2)}(\beta, c),
\]

where $P_n^{(1/2)}(\beta, c)$ is the $n$th Pollaczek polynomial, cf. [5], of order $1/2$ for $(-\infty, \infty)$.

References


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