ABSOLUTE F_σ SPACES

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1. **Introduction.** All spaces referred to in this paper are assumed to be *metric* (or, more accurately, metrizable); metrics are denoted by symbols like ρ. A space X is said to be of *absolute* Borel class α (where α is a countable ordinal) if, whenever X is a subspace of a space Y, X is Borel of class α in Y.[1] Clearly, if X is of absolute Borel class α, then so is every space homeomorphic to X. For all but the simplest Borel classes, such spaces are characterized by the well-known theorem (see e.g., [6, p. 339]):

(1) X is of absolute Borel class α if (and only if) X is of Borel class α in some complete (metric) space.

But this theorem applies only to the classes G_δ and beyond ($F_{\sigma\delta}$, $G_{\delta\sigma}$, etc., i.e., $\alpha \geqq 2$). The only nontrivial case remaining is that of the absolute F_σ spaces. It is well known (and elementary) that a *separable* space is an absolute F_σ if and only if it is σ-compact (i.e., the union of countably many compact spaces). For nonseparable spaces this fails (e.g., an uncountable discrete set is an absolute F_σ but not σ-compact); a valid generalization (Theorem 2 below) is the main object of this paper. We shall also characterize the spaces which are absolutely the difference between two closed (or, equivalently, open) sets; here the known result for separable spaces needs no alteration in the general case (Theorem 1), though the proof does. The more complicated types—e.g., the spaces which are absolutely the difference between two F_σ (or, equivalently, G_δ) sets—are covered by the standard theory [6, p. 339]; the analogue of (1) applies to them. For completeness, we mention that the only absolutely open space is the empty set; the absolutely closed spaces are, of course, the compact spaces.

2. **Absolute $F\cap G$ spaces.** We state as a lemma the following fundamental result of Hausdorff:

LEMMA 1. *Given a closed subspace A of a (metric) space X, and given any metric ρ on A (compatible with the topology of A), there exists a metric ρ^* on X (compatible with the topology of X) which, restricted to A, coincides with ρ.*

For proofs, see [2; 3; 5].

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[1] For simplicity of statement, we are ignoring here the distinction between additive and multiplicative Borel classes.

THEOREM 1. *A space X is absolutely the difference between two closed sets if and only if it is locally compact.*

One implication is well known (and applies to any Hausdorff spaces, not necessarily metric). If X is a locally compact subset of a space Y, then X is open relative to $\overline{X}$ (see e.g., [4, p. 99]), and is thus the intersection of an open set and a closed set in Y—or, equivalently, is the difference between two closed (or open) sets.

Conversely, suppose X is *not* locally compact. There is a point $a \in X$ such that, for all $\epsilon > 0$, the closure of the neighborhood $S(a, \epsilon) = \{x \mid x \in X, \rho(a, x) < \epsilon\}$ fails to be compact. Then $\overline{S}(a, 1)$ contains a countably infinite discrete set $B_1 = \{b_{1n}\}$ $(n = 1, 2, \cdots)$ with no limit point in X. B_1 is clearly closed in X; further, we may assume $a \notin B_1$ (otherwise replace B_1 by $B_1 - (a)$). Thus, on writing $\delta_1 = \rho(a, B_1)$, we have $0 < \delta_1 \leqq 1$. Similarly $\overline{S}(a, \delta_1/2)$ contains an infinite discrete (closed) set $B_2 = \{b_{2n}\}$ $(n = 1, 2, \cdots)$ with $0 < \rho(a, B_2) = \delta_2 \leqq \delta_1/2$. Proceeding in this way we obtain, for each $m = 1, 2, \cdots$, a closed infinite discrete set $B_m = \{b_{mn}\}$ $(n = 1, 2, \cdots)$ such that $0 < \rho(a, B_m) = \delta_m \leqq \delta_{m-1}/2$ $(m > 1)$. Put $A = (a) \cup \bigcup B_m$; clearly A is closed.

Now consider the subset A^* of the plane consisting of the points $a^* = (0, 0)$, $b^*_{mn} = (1/m, 1/(m+n-1))$ $(m, n = 1, 2, \cdots)$. The mapping $f: A \to A^*$ defined by $f(a) = a^*$, $f(b_{mn}) = b^*_{mn}$, is easily verified to be a homeomorphism of A onto A^*. Thus, if d denotes the Euclidean distance in the plane, and if we put $\rho'(x, y) = d(f(x), f(y))$, then ρ' is a metric on A (compatible with the topology of A). By Lemma 1 there is an extension ρ^* of ρ' to a metric on X. Let Y denote the completion of X in this metric.

For each fixed m, the sequence $\{b_{mn}\}$ $(n = 1, 2, \cdots)$ is a Cauchy sequence in X, in the metric ρ^*, and it therefore converges to some $b_m \in Y$. We extend f to f^* by defining $f^*(b_m) = \lim_{n \to \infty} b^*_{mn} = (1/m, 0) = b^*_m$ say; since f is an isometry, so is f^*. The points b_m $(m = 1, 2, \cdots)$ are therefore all distinct, and converge to a as $m \to \infty$.

Suppose now that X is the difference between two closed sets in Y; thus $X = F \cap G$ where F is closed and G is open in Y. Thus, for each m, $b_m \in \overline{X} \subset F$. Again, $b_m \to a \in X \subset G$; hence, for some M, $b_M \in G$. That is, $b_M \in F \cap G = X$; and the sequence b_{Mn} $(n = 1, 2, \cdots)$ converges to b_M in X. But this contradicts the choice of B_M as a discrete subset of X.

3. **Absolute F_σ spaces.** We recall that the *weight* of a space X is the least cardinal of any basis of open sets of X. Since we are dealing with metric spaces only, it is also the least cardinal of any dense subset of X.

LEMMA 2. *If every point of a (metric) space X has a neighborhood of weight $\leqq k$, X can be expressed as a union of disjoint open sets X_λ, each of weight $\leqq k$.*

When $k=\aleph_0$ this is a well-known theorem of Alexandroff [1, p. 300]; another proof is in [7]. Actually this special case is all we shall need; but it may be of interest to sketch a simple proof of the general result. We may of course assume k infinite. Being paracompact, X has a locally finite covering $\{U_\alpha\}$ by open sets, each of weight $\leqq k$. Write $U_\alpha \sim U_\beta$ to mean that there is a finite sequence $U_\alpha = U_{\gamma_0}$, $U_{\gamma_1}, \cdots, U_{\gamma_n} = U_\beta$, such that $U_{\gamma_i} \cap U_{\gamma_{i-1}} \neq \emptyset$ $(i=1, 2, \cdots, n)$. (That is, U_α and U_β belong to the same component of the nerve of the covering.) This defines an equivalence relation; suppose the distinct equivalence classes are $\{\mathcal{C}_\lambda | \lambda \in \Lambda\}$. For each λ, put $X_\lambda = \bigcup\{U_\alpha | U_\alpha \in \mathcal{C}_\lambda\}$. The sets X_λ are disjoint, open and cover X; and, because each $\mathcal{C}_\lambda$ is easily seen to have cardinal $\leqq k$, each X_λ has weight $\leqq k$.

LEMMA 3. *Let k be an infinite cardinal, not the sum of $\aleph_0$ smaller cardinals. If every nonempty open subset of a nonempty complete (metric) space Y has weight $\geqq k$, Y contains a closed subset homeomorphic to the product $B(k)$ of $\aleph_0$ discrete sets, each of cardinal k.*[2]

This is a slight generalization of [8, Lemma 3.1], and is proved by the same argument.

LEMMA 4. *A necessary and sufficient condition that X be an absolute F_σ is that, in every metric on X (compatible with its topology), X is σ-complete.*

If ρ is any metric on an absolute F_σ space X, then X is F_σ in its completion $\tilde{X}$ in the metric ρ. Thus $X = \bigcup X_n$ $(n=1, 2, \cdots)$, where X_n is closed in $\tilde{X}$ and therefore complete with respect to ρ. Conversely, if the condition is satisfied, let X be a subspace of a space Y with metric ρ. Then ρ restricted to X gives a metric on X, for which $X = \bigcup X_n$ with X_n complete. Hence X_n is closed, and X is F_σ, in Y.

LEMMA 5. *Every closed subspace of an absolute F_σ is an absolute F_σ.*

Let A be closed in an absolute F_σ set X. By Lemma 1, every metric ρ on A can be extended to a metric ρ^* on X. By Lemma 4, $X = \bigcup X_n$ where X_n is complete in ρ^*. Hence $A = \bigcup(A \cap X_n)$ where $A \cap X_n$ is complete in the metric ρ. By Lemma 4 again, A is an absolute F_σ.

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[2] The conclusion is in fact true even when k is the sum of $\aleph_0$ smaller cardinals, providing $k > \aleph_0$; but this needs further argument. We shall need Lemma 3 only when $k = \aleph_1$.

LEMMA 6. *Every F_σ subspace of an absolute F_σ is an absolute F_σ.*

Let $A=\bigcup A_n$ where A_n is closed in an absolute F_σ space X. If $A\subset Y$, Lemma 5 shows that each A_n is F_σ in Y; say $A_n=\bigcup F_{nm}$ $(m=1, 2, \cdots)$ where F_{nm} is closed in Y. Thus $A=\bigcup_{m,n} F_{nm}$ and is F_σ in Y.

LEMMA 7. *If X is a complete absolute F_σ, then $X=\bigcup X_n$ where each X_n is a locally separable F_σ subset of X $(n=1, 2, \cdots)$.*

Define $K_0=X$, $U_0=\emptyset$. When K_β, U_β have been defined for all ordinals $\beta<\alpha$, we proceed as follows.

(a) If α has a predecessor, α^-, put $U_\alpha=$union of all relatively open separable subsets of K_{α^-}, $K_\alpha=K_{\alpha^-}-U_{\alpha^-}$.

(b) If α is a limit ordinal, put $U_\alpha=\emptyset$, $K_\alpha=\bigcap\{K_\beta|\beta<\alpha\}$.

Clearly U_α is open relative to K_{α^-}, in case (a); hence the sets K_α form a decreasing transfinite sequence of closed subsets of X. For some sufficiently large ordinal α^* (not necessarily countable, of course), we have $K_{\alpha^*}=K_{\alpha^*+1}=\cdots=K$, say. We shall show that $K=\emptyset$. In fact, since $U_{\alpha^*+1}=\emptyset$ here, the construction shows that each nonempty relatively open subset of K must be nonseparable, i.e., of weight $\geqq\aleph_1$. By Lemma 3, if $K\neq\emptyset$, K has a closed subset homeomorphic to $B(\aleph_1)$. This in turn has a closed subset J homeomorphic to $B(\aleph_0)$—i.e., to the space of irrational numbers. Being closed in X, J must be an absolute F_σ, by Lemma 5. But a well-known consequence of Baire's theorem is that J is not F_σ in the real line. Hence $K=\emptyset$.

In what follows, α runs over all ordinals $<\alpha^*$. Since $K=\emptyset$, it readily follows that $\bigcup U_\alpha=X$. Define, for $n=1, 2, \cdots$, $V_{\alpha n}=\{x|x\in U_\alpha,\ \rho(x, K_\alpha)\geqq 1/n\}$. (We make the convention that $\rho(x, \emptyset)=\infty$.) Then $V_{\alpha n}$ is the intersection of the closed set $\{x|\rho(x, K_\alpha)\geqq 1/n\}$ with U_α, which in turn is either $\emptyset$ or the intersection of K_{α^-} and an open subset of X. Hence $V_{\alpha n}$ is F_σ in X. Further, one easily verifies that $\rho(V_{\alpha n}, V_{\beta n})\geqq 1/n$ if $\alpha\neq\beta$. Thus, if we put $X_n=\bigcup_\alpha V_{\alpha n}$, X_n is also F_σ in X. Clearly $\bigcup X_n=\bigcup_{\alpha,n} V_{\alpha n}=\bigcup U_\alpha=X$. All that remains to be proved is that X_n is locally separable. Now, given $x\in X_n$, we have $x\in V_{\alpha n}$ for some α, and then $S(x, 1/n)\cap X_n\subset U_\alpha$. But U_α is, by construction, a union of open separable subsets of K_{α^-} (for $U_\alpha\neq\emptyset$ here); hence, for some $\epsilon>0$, $S(x, \epsilon)\cap K_{\alpha^-}$ is separable. If $\delta=\min(\epsilon, 1/n)$, then $S(x, \delta)\cap X_n\subset S(x, \delta)\cap U_\alpha\subset S(x, \epsilon)\cap K_{\alpha^-}$, and is therefore also separable.

LEMMA 8. *If X is a locally separable absolute F_σ, X is σ-locally compact.*

By Lemma 2 we have $X=\bigcup X_\lambda$ where the sets X_λ are disjoint, open

and separable. Each X_λ can be imbedded in a copy H_λ of the Hilbert cube, and we may assume that the sets H_λ are disjoint and of diameter 1. We extend the metrics on the separate sets H_λ to a metric ρ on $H=\bigcup H_\lambda$ by taking $\rho(x, y)=1$ whenever $x\in H_\lambda$, $y\in H_\mu$, and $\lambda\neq\mu$. There is now an obvious homeomorphism of X onto a subset X' of H, and we have $X'=\bigcup F_n'$ $(n=1, 2, \cdots)$ where F_n' is closed in H. Clearly F_n' is locally compact; in fact, if $x\in F_n'\cap H_\lambda$, $F_n'\cap\bar{S}(x, 1/2)\subset H_\lambda$ and so is a compact neighborhood of x in F_n'. Thus $X=\bigcup F_n$ where F_n is locally compact.

THEOREM 2. *A necessary and sufficient condition for a (metric) space to be an absolute F_σ, is that it be σ-locally compact.*

Suppose X is an absolute F_σ. By Lemma 4, $X=\bigcup X_m$ $(m=1, 2, \cdots)$ where X_m is complete (in the metric on X), and so closed. By Lemma 5, X_m is an absolute F_σ. Hence, by Lemma 7, $X_m=\bigcup X_{mn}$ $(n=1, 2, \cdots)$ where X_{mn} is locally separable and F_σ in X. By Lemma 6, X_{mn} is an absolute F_σ; by Lemma 8 it follows that $X_{mn}=\bigcup X_{mnp}$ $(p=1, 2, \cdots)$ where X_{mnp} is locally compact. Thus finally

$$X = \bigcup \{X_{mnp} \mid m, n, p = 1, 2, \cdots\},$$

proving X σ-locally compact.

Conversely, suppose X is a σ-locally compact subset of a space Y, say $X=\bigcup X_n$ $(n=1, 2, \cdots)$ where X_n is locally compact. By Theorem 1, X_n is the difference between two closed subsets of Y, and is therefore F_σ in Y; say $X_n=\bigcup F_{nm}$ $(m=1, 2, \cdots)$ where F_{nm} is closed in Y. Then $X=\bigcup_{m,n} F_{nm}$ and is F_σ in Y.

REMARK. Since every separable locally compact space is σ-compact, Theorem 2 includes the classical result that for a *separable* space to be an absolute F_σ it is necessary and sufficient that it be σ-compact.

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