DIRECT DECOMPOSITIONS OF LATTICES OF CONTINUOUS FUNCTIONS

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If $X$ is a topological space and if $K$ is a chain equipped with its order topology, then we denote by $C(X, K)$ the lattice of all continuous functions from $X$ to $K$. If $X$ is the union of two disjoint open-and-closed subsets $X_1$ and $X_2$, then it is clear that $C(X, K)$ is isomorphic to the direct product of the lattices $C(X_1, K)$ and $C(X_2, K)$. In Theorem 2 of [2], Kaplansky proves the following converse:

Theorem A (Kaplansky). If $X$ is compact, if $K$ has neither a first nor a last element, and if $C(X, K)$ is isomorphic to the direct product of two lattices $L_1$ and $L_2$, then $X$ is the union of disjoint open-and-closed subsets $X_1$ and $X_2$ having the property that $L_i$ is isomorphic to $C(X_i, K)$ ($i = 1, 2$).

A technique for removing the stated hypothesis on $K$ is outlined in §6 of [2]. The validity of Theorem A for noncompact spaces, however, is left as an open question in [2]. In this note we shall remove from Theorem A both the hypothesis on $K$ and the compactness hypothesis on $X$. At the same time, we shall show that a direct decomposition of merely a sublattice of $C(X, K)$ (satisfying a very mild condition) is enough to ensure a corresponding decomposition of $X$ (Theorem B below). The sublattices that we find adequate for this purpose are described as follows (cf. the concluding remark of this note):

Definition. A sublattice $L$ of $C(X, K)$ will be called adequate in case for each $x \in X$ there exist functions $f, g \in L$ such that $f(x) < g(x)$.

For example, if $L$ is a sublattice of $C(X, K)$ that contains at least two distinct constant functions, then obviously $L$ is adequate.

By a prime ideal of a lattice $L$ we mean a nonempty proper subset $P$ of $L$ such that (i) if $a, b \in P$, then $a \vee b \in P$ and (ii) $a \wedge b \in P$ if and only if $a \in P$ or $b \in P$; a dual prime ideal is the complement of a prime ideal (see e.g. [1]). We require the following readily verified fact (cf.

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1 If $K$ is the chain $R$ of real numbers, then (as observed in [2, p. 621]) a reduction to the compact case is possible via the Stone-Čech compactification (of a suitable completely regular space). One should note, however, that this device yields Theorem A (for $X$ arbitrary) with $C(X, K)$ and $C(X_1, K)$ replaced, respectively, by the lattices $C^*(X, R)$ and $C^*(X_1, R)$ of bounded real-valued continuous functions on $X$ and $X_1$.

2 Our proof is a modification of Kaplansky's original argument. No separation properties are required of $X$.

631
If $L_1$ and $L_2$ are lattices and if $P$ is a prime ideal of the direct product $L_1 \times L_2$, then either $P = P_1 \times L_2$ for some prime ideal $P_1$ of $L_1$ or $P = L_1 \times P_2$ for some prime ideal $P_2$ of $L_2$.

If $Y$ is a subset of $X$ and if $f \in C(X, K)$, then $f\mid Y$ denotes the restriction of $f$ to $Y$. If $L$ is a sublattice of $C(X, K)$, then we set

$L_Y = \{ f\mid Y : f \in L \}$.

It is clear that $L_Y$ is a sublattice of $C(Y, K)$.

We can now state the following result:

**Theorem B.** Let $X$ be a topological space, let $K$ be a chain equipped with its order topology, and let $L$ be an adequate sublattice of $C(X, K)$. If $L$ is isomorphic to the direct product of two lattices $L_1$ and $L_2$, then $X$ is the union of disjoint open-and-closed subsets $X_1$ and $X_2$ having the property that $L_i$ is isomorphic to $L_{X_i}$ (i = 1, 2). (The isomorphisms involved are described explicitly below.) Moreover, $X_1$ is nonempty if and only if $L_i$ has at least two distinct elements.

**Proof.** If $x \in X$ and $f \in L$, we set

$P_x(f) = \{ g \in L : g(x) \leq f(x) \}$

and

$P^*(f) = \{ g \in L : g(x) \geq f(x) \}$.

It is clear that $P_x(f)$ (resp. $P^*(f)$) is a prime (resp. dual prime) ideal of $L$ provided only that it is a proper subset of $L$. The adequacy of $L$ then ensures that, in any event, either $P_x(f)$ is a prime ideal of $L$ or $P^*(f)$ is a dual prime ideal of $L$.

We choose now an isomorphism $\delta$ from $L$ onto $L_1 \times L_2$ and a fixed element $\delta \in L$. Denote by $\emptyset_1$ (resp. $\emptyset_2$) the collection of all prime ideals $P$ of $L$ such that $\delta(P)$ is of the form $P_1 \times L_2$ (resp. $L_1 \times P_2$), with $P_i$ a prime ideal of $L_i$. For $i = 1, 2$, denote by $X_i$ the set of all points $x \in X$ such that either $P_x(k) \in \emptyset_i$ or $L - P^*(k) \in \emptyset_i$. Then it is easily seen that $X_1$ and $X_2$ are disjoint and that $X = X_1 \cup X_2$. Moreover, if $y$ is in the closure of $X_i$, then

$\cap \{ P_x(k) : x \in X_i \} \subseteq P_y(k)$

and

$\cap \{ P^*(k) : x \in X_i \} \subseteq P^*(k)$,

from which it follows that $y \in X_i$. Thus both $X_1$ and $X_2$ are open-and-closed.

Now let $\pi_i$ be the projection of $L_1 \times L_2$ onto $L_i$, and consider the
mapping \( \phi_i = \pi_i \circ \delta \) from \( L \) onto \( L_i \). Let \( f, g \in L \) and suppose that \( \phi_i(f) \leq \phi_i(g) \) but that \( f(x) > g(x) \) for some \( x \in X_1 \). Then \( P_a(g) \) is a prime ideal of \( L \) that contains \( g \) but not \( f \). If \( P_a(k) \neq L \), then, since \( P_a(g) \cap P_a(k) \) contains a prime ideal of \( L \) (namely, \( P_a(g \wedge k) \)), \( P_a(g) \) must map onto \( P_a \times L_2 \) for some prime ideal \( P_1 \) of \( L_1 \). But then \( \phi_i(f) \in P_1 \) so that \( f \in P_a(g) \), a contradiction. Moreover, if \( P_a(k) \neq L \), then a dual argument again yields a contradiction. Arguing similarly for \( X_2 \), we therefore conclude that

\[
(1) \quad \phi_i(f) \leq \phi_i(g) \text{ implies } f \upharpoonright X_i \leq g \upharpoonright X_i \quad (i = 1, 2).
\]

Now suppose, on the other hand, that \( f \upharpoonright X_1 \leq g \upharpoonright X_1 \) but that \( \phi_i(f) \nleq \phi_i(g) \). Since \( L_i \) is distributive, Zorn’s lemma provides a prime ideal \( P_1 \) in \( L_1 \) that contains \( \phi_i(g) \) but not \( \phi_i(f) \). Let \( P \) be the prime ideal in \( L \) that maps onto \( P_1 \times L_2 \). Then \( g \in P \) and \( f \notin P \). Let \( h = \delta^{-1}(\phi_i(f), \phi_i(g)) \) so that \( h \in P \). Now \( \phi_i(h) = \phi_i(g) \) and therefore, by (1), \( h \upharpoonright X_2 = g \upharpoonright X_2 \). But then \( f \upharpoonright h \leq g \) so that \( f \wedge h \in P \), a contradiction. Using a similar argument for \( \phi_2 \), we thus obtain

\[
(2) \quad f \upharpoonright X_i \leq g \upharpoonright X_i \text{ implies } \phi_i(f) \leq \phi_i(g) \quad (i = 1, 2).
\]

We conclude from (2) that \( \psi_i : f \upharpoonright X_i \mapsto \phi_i(f) \) is a well-defined order-preserving map from \( L_{X_i} \) onto \( L_i \). Moreover, by (1), \( \psi_i \) is one-to-one and \( \psi_i^{-1} \) is also order-preserving. Hence \( \psi_i \) is an isomorphism.

Using the adequacy of \( L \), note finally that \( X_i \) is nonempty if and only if \( L_{X_i} \) has at least two distinct elements. Since \( L_i \) is isomorphic to \( L_{X_i} \), the last assertion of the theorem is immediate, and the proof is complete.

**Remark 1.** Let \( \delta \) and \( \pi_i \) be as above and let \( \lambda_i \) be an arbitrary isomorphism from \( L_i \) into \( L_i \times L_2 \) such that \( \pi_i \circ \lambda_i \) is the identity on \( L_i \). Let \( \rho_i \) be the restriction homomorphism \( f \mapsto f \upharpoonright X_i \) from \( L \) onto \( L_{X_i} \). The proof of Theorem B shows that \( \rho_i \circ \delta^{-1} \circ \lambda_i \) is an isomorphism from \( L_i \) onto \( L_{X_i} \) and that, for each \( f \in L \),

\[
(\rho_i \circ \delta^{-1} \circ \lambda_i)(\pi_i(\delta(f))) = \rho_i(f) \quad (i = 1, 2).
\]

**Remark 2.** If \( P_a(k) \) is always a prime ideal of \( L \) (and this is the case, for example, if \( K \) has no last element and if \( L \) contains every constant function on \( X \) to \( K \)), then the proof of Theorem B admits the following simplification: Ignoring \( P_a(k) \), we can take \( X_i \) to be the set of all \( x \in X \) such that \( P_a(k) \in \mathfrak{P}_i \) (cf. the proof of Theorem 2 of [2]).

In the following corollary, \( C*(X, K) \) denotes the sublattice of \( C(X, K) \) consisting of all bounded continuous functions from \( X \) to \( K \).

**Corollary.** Let \( X \) and \( K \) be as before. If \( C(X, K) \) (resp. \( C*(X, K) \)) is isomorphic to the direct product of two lattices \( L_1 \) and \( L_2 \), then \( X \) is the
union of disjoint open-and-closed subsets $X_1$ and $X_2$ having the property that $L_i$ is isomorphic to $C(X_i, K)$ (resp. $C^*(X_i, K)$) ($i = 1, 2$).

Proof. If $K$ consists of a single element, we can take $X_1 = X$ and $X_2 = \emptyset$; the result is then a consequence of the fact that $C(\emptyset, K) = \{ \emptyset \}$. If $K$ has at least two elements, then both $C(X, K)$ and $C^*(X, K)$ are adequate, and the result follows immediately from the theorem.

Remark 3. The following question remains open: What are necessary and sufficient conditions on a sublattice $L$ of $C(X, K)$ in order that a direct decomposition of $L$ be reflected in a corresponding decomposition of $X$? In any case, the hypothesis of adequacy cannot simply be deleted. To see this, let $R$ be the chain of real numbers, let $i: R \to R$ be the identity mapping, and let $L$ be the (nonadequate) sublattice of $C(R, R)$ generated by $i$ and $-i$. If $K$ is any chain with exactly two elements, then $L$ is isomorphic to $K \times K$, but there is no corresponding decomposition of $R$.

References


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