ON COMPLETE BERGMAN METRICS

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1. In [3] we gave a sufficient condition for the Bergman metric to be complete. We shall give here a slightly modified condition for the completeness. To state our result more explicitly, we shall recall definitions given in [3].

2. Let $M$ be an $n$-dimensional complex manifold, $F$ the Hilbert space of holomorphic $n$-forms $f$ on $M$ such that

$$(n-1)^2 \int_M f \wedge \bar{f} < \infty.$$ 

Let $h_0, h_1, h_2, \cdots$ be an orthonormal basis for $F$. The kernel form $K$ of Bergman is defined by

$$K = \sum h_i \wedge \bar{h}_i.$$ 

(Strictly speaking, one should put $(-1)^{n+1}$ in front of $\sum$; but this is not essential in the following discussion.)

Suppose $F$ is ample in the following sense:

(A.1). For every $z$ in $M$, there exists an $f$ in $F$ which does not vanish at $z$.

(A.2). For every holomorphic vector $Z$ at $z$, there exists an $f$ in $F$ such that $f$ vanishes at $z$ and $Z(f^*) \neq 0$, where $f = f^* dz_1 \wedge \cdots \wedge dz_n$ with respect to a local coordinate system $z_1, \cdots, z_n$ of $M$.

If $F$ satisfies the conditions (A.1) and (A.2), then the Bergman metric $ds^2$ is defined by

$$ds^2 = \sum \frac{\partial^2 \log K^*}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b$$

where $K = K^* dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$;

If $M$ is a bounded domain in $\mathbb{C}^n$, then $F$ is ample and the Bergman metric is defined; this is of course the case originally considered by Bergman [1].

3. Consider now the following additional condition

(C). For every infinite sequence $S$ of points of $M$ which has no adherent point in $M$ and for every $f$ in $F$, there exists a subsequence $S'$ of $S$ such that

$$\lim_{s' \to s} (f \wedge \bar{f})/K = 0.$$ 

Received by the editors June 7, 1961.

1 This work has been supported by N.S.F. Grant 10375.

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Then we know that

(i). A complex manifold \( M \) satisfying (C) is complete with respect to the Bergman metric [3]. (I conjecture that the converse is true.)

(ii). A bounded domain in \( C^n \) which is complete with respect to the Bergman metric is a domain of holomorphy. The converse is not true [2].

(iii). Every domain of holomorphy can be approximated by an increasing sequence of analytic polyhedrons.

(iv). Every bounded analytic polyhedron in \( C^n \) satisfies (C) [3].

The above four statements show that the three concepts "holomorph-convexity," "metric completeness" and "(C)" are closely related to each other. Concerning (ii), it is not known whether a complex manifold which is complete with respect to the Bergman metric is necessarily holomorph-convex. It is also unknown whether (C) implies the holomorph-convexity for a manifold. For the proof of (ii), Bremermann makes use of the ambient space \( C^n \) which is not available in the case of an abstract complex manifold. Let \( M \) be a bounded domain of holomorphy in \( C^n \) and let \( A(M) \) be the intersection of all the domains of holomorphy \( G \) containing the closure of \( M \). According to Sommer-Mehring [4], the assumption \( A(M) = M \) implies that the kernel function can not be continued outside \( M \). It is very likely that \( A(M) = M \) implies the completeness with respect to the Bergman metric.

4. We shall now consider the following condition

\( (C') \). Let \( F' \) be a (fixed) dense subset of the Hilbert space \( F \). For every infinite sequence \( S \) of points of \( M \) which has no adherent point in \( M \) and for every \( f \) in \( F' \), there exists a subsequence \( S' \) of \( S \) such that

\[
\lim_{S'} (f \wedge \bar{f})/K = 0.
\]

We shall prove

**Theorem.** If a complex manifold \( M \) with Bergman metric satisfies \( (C') \) (for some dense subset \( F' \) of \( F \), then \( M \) is complete with respect to the Bergman metric.

The proof is a slight modification of the argument in our previous paper [3, p. 284] and we shall use the same notations as in [3]. Let \( H \) be the dual space of \( F \) and \( P(H) \) the projective space of complex 1-dimensional subspaces of \( H \); the dimension of \( P(H) \) is possibly infinite. In [3, see pp. 280–282]], we defined a natural Kaehler metric \( d\sigma^2 \) on \( P(H) \) and proved the metric completeness of \( P(H) \). The natural imbedding \( j: M \to P(H) \) defined in [3] is isometric in the sense of differential geometry, i.e., \( j^*(d\sigma^2) = ds^2 \). The distance between two
points of $M$ (resp. $P(H)$) is the greatest lower bound of the lengths of the piecewise differentiable curves joining them in $M$ (resp. $P(H)$). It follows that, for every pair of points $z$ and $z'$ of $M$, the distance between $j(z)$ and $j(z')$ with respect to $da^2$ does not exceed the one between $z$ and $z'$ with respect to $ds^2$. Assuming that $M$ is not complete, let $S$ be a Cauchy sequence in $M$ which has no limit point in $M$. Then $j(S)$ is a Cauchy sequence in $P(H)$. By the completeness of $P(H)$, $j(S)$ has a limit point, say $x_0$, in $P(H)$. By a proper choice of basis in $H$, we may assume that $x_0$ is represented by a point $\xi_0 = (1, 0, 0, \cdots)$ of $H$. Take the dual basis $h_0, h_1, h_2, \cdots$ in $F$. Let $f$ be an element of $F'$. Then

$$f = \sum_{j=0}^{\infty} a_j h_j, \quad a_j \in \mathbb{C}.$$  

For any $z$ in $M$, $j(z)$ is represented by the point of the unit sphere in $H$ whose homogeneous coordinates are given by

$$(h_0(z) \wedge \bar{h}_0(z)/K(z, \bar{z}), \quad h_1(z) \wedge \bar{h}_1(z)/K(z, \bar{z}), \quad h_2(z) \wedge \bar{h}_2(z)/K(z, \bar{z}), \cdots).$$

Hence, $\lim_{S'} (f \wedge \bar{f})/K = |a_0|^2$. Let $S'$ be any subsequence of $S$. Since $j(S')$ and $j(S)$ have the same limit point,

$$\lim_{S'} (f \wedge \bar{f})/K = |a_0|^2.$$  

In order that the condition (C') holds, $a_0$ must be zero. That would imply that $F'$ is orthogonal to $h_0$, contradicting the assumption that $F'$ is dense in $F$. Q.E.D.

**Corollary.** Let $M$ be a bounded domain in $\mathbb{C}^n$. If the polynomials are dense in $F$ and if the Bergman's kernel function goes to infinity at every boundary point of $M$, then $M$ is complete with respect to the Bergman metric.

**Bibliography**