REPRESENTATIONS OF BANACH SPACES

Banach and Mazur proved that every separable Banach space $B$ can be represented as the space $C(M)$ of continuous real functions on a compact metric space $M$. Since $M$ is the continuous image of the Cantor set $K$, $C(M)$ can be imbedded in $C(K)$, and since functions in $C(K)$ can be extended preserving norm to functions over $I^1$, they conclude that $B$ can be represented as a subspace of $C(I^1)$.

If $B$ is not separable, it can be represented as $C(H)$, where $H$ is compact Hausdorff. A compact Hausdorff space is the continuous image of some totally disconnected compact Hausdorff space $T$—for example, give the space the discrete topology, and let $T$ be its Stone-Čech compactification. It follows that $B$ is isomorphic to a subspace of $C(T)$. If $T$ could be given a linear order inducing the same topology, we could fill in the missing intervals and obtain a compact con-

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NORTHWESTERN UNIVERSITY

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G. S. YOUNG
nected ordered space $S$, and so represent $B$ as a subspace of $C(S)$. I cannot have been the only one who observed this, and failed in ordering $T$.

Mardesić and Papić have proved the unexpected and elegant result that if $\Pi X_a$ is a product of two or more nondegenerate continua, and there is a continuous map of a compact ordered continuum onto $\Pi X_a$, then each $X_a$ is metric. From this it follows, e.g., that if $L$ is the “long interval,” obtained by inserting open intervals between consecutive points of the set of ordinals not greater than the first uncountable ordinal, then $L \times I^1$ is not the continuous image of any compact ordered space, $T$.

For suppose that there is a map $f: T \rightarrow L \times I^1$, onto. Let $a_1, a_2$ be two points of $T$ with no point between them. Let $f(a_j) = (l_j, t_j)$, $j = 1, 2$, and suppose $l_1 \leq l_2$ in $L$. The set $L$ contains an “interval” $l_1 l_2$, possibly a point, and in $I^1$ there is an interval $t_1 t_2$, also possibly a point. The set $X = l_1 l_2 \times t_1 \cup t_2 l_2$ is a continuum ordered by separation. Between $a_1$ and $a_2$ insert a copy $X_{12}$ of $X - f(a_1) - f(a_2)$ so that the natural map of $a_1 X_{12} a_2$ onto $X$ is a homeomorphism. Having done this for each pair $a_1, a_2$, we obtain a compact connected ordered space $T^*$. We define $f^*: T^* \rightarrow L \times I^1$ to agree with $f$ on $T$ and to be the natural map on each set $X_{12}$. It is easily seen that $f^*$ is continuous. But the Mardesić-Papić result then implies that $L$ is metric, a contradiction. There are obvious generalizations. A corollary is that there exist compact totally disconnected Hausdorff spaces that cannot be ordered.

The algebra of $C(L \times I^1)$ must be quite different from that of $C(I^1 \times I^1)$. It would be interesting to see how this property manifests itself algebraically.