SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

SHORT PROOF OF A THEOREM OF WINTNER ON ASYMPTOTIC INTEGRATIONS OF THE ADIABATIC OSCILLATOR

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The theorem below on real functions was proved by Wintner [2] and generalized by Levinson [1]. "Solution" to an nth order system of differential equations will mean a set of n absolutely continuous functions satisfying the given system almost everywhere.

THEOREM. *Let $g>\rho>-1$ (ρ a constant) be a function of bounded variation on $\{x: 0\leqq x<\infty\}$. The differential equation $-y''=(f')^2y$, where $'=d/dx$ and $f=\int_0^x(1+g)^{1/2}$, has solutions y_1 and y_2 such tha, $y_1\to\sin f$ and $y_2\to\cos f$ while $y_1'\to f'\cos f$ and $y_2'\to=-f'\sin f$ as $x\to\infty$*

PROOF. Denote by J the set on which g is differentiable, and put $F=f''/f'$. As suggested by the case where f' equals a constant, functions r and θ are introduced to seek a solution such that

$$(1)\quad y(x) = r(x)\sin(f(x)+\theta(x));\quad y'(x) = r(x)f'(x)\cos(f(x)+\theta(x)).$$

This requires that

$$(2)\qquad \sin(f(x)+\theta(x))\frac{r'(x)}{r(x)}+\cos(f(x)+\theta(x))\theta'(x) = 0,$$

while substitution into $-y''=(f')^2y$ yields

$$(3)\qquad \begin{aligned}\cos(f(x)+\theta(x))\frac{r'(x)}{r(x)} &- \sin(f(x)+\theta(x))\theta'(x)\\ &= -F(x)\cos(f(x)+\theta(x)) \qquad (x\in J)\end{aligned}$$

Solving (2) and (3) for θ' and r'/r gives

$$(4)\qquad \begin{aligned}2\theta'(x) &= F(x)\sin 2(f(x)+\theta(x));\\ \frac{r'(x)}{r(x)} &= -F(x)\cos^2(f(x)+\theta(x)) \qquad (x\in J).\end{aligned}$$

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Since $\int_0^\infty |F| < \infty$, the first equation in (4) has a solution, say θ_1, such that $\theta_1 \to 0$ as $x \to \infty$. Put $r_1 = \exp - \int_\infty^x F \cos^2 (f + \theta_1)$. Then r_1 and θ_1 constitute a solution of (4). Set $y_1 = r_1 \sin (f + \theta_1)$. One can construct y_2 similarly.

References

1. N. Levinson, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J. 15 (1948), 111–126.

2. A. Wintner, *Asymptotic integrations of the adiabatic oscillator*, Amer. J. Math. 69 (1947), 251–272.

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REMARK ON $\pi(x) = o(x)$

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In Hardy and Wright [1], Landau [2], and Prachar [3] there are proofs of $\pi(x) = o(x)$. The purpose of this paper is to provide a simpler proof.

THEOREM. $\pi(x) = o(x)$.

PROOF. Let M_r denote the product of the first r primes. Given any positive real number $x \geqq 2$, it is clear that there exist unique positive integers k and r, with $1 \leqq k \leqq p_{r+1} - 1$, such that

$$kM_r \leqq x < (k+1)M_r. \tag{1}$$

For any x satisfying (1) we must also have

$$\pi(x) \leqq (k+1)\phi(M_r) + r \tag{2}$$

(where $\phi(M_r)$ is the totient of M_r). From (1) and (2) we obtain

$$0 \leqq \frac{\pi(x)}{x} \leqq \frac{(k+1)\phi(M_r)}{kM_r} + \frac{r}{kM_r}. \tag{3}$$

Since $kM_r \geqq kr! \geqq r!$ and $(k+1)/k \leqq 2$, we replace (3) by

$$0 \leqq \frac{\pi(x)}{x} \leqq 2 \prod_{i=1}^{r} (1 - p_i^{-1}) + \frac{1}{(r-1)!}. \tag{4}$$

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