ROOTS OF SCALAR OPERATORS

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Introduction. The existence of normal roots of normal operators is a well-known consequence of the spectral theorem. Very simple examples show however that normal operators may have roots which are not normal (in fact the identity operator has non-normal roots). In this paper we will show that the invertible scalar operators on a Banach space possess only scalar operators as roots.

Preliminaries. The term operator will be used to mean a bounded linear transformation of a Banach space into itself. By a spectral measure on a Banach space $\mathfrak{X}$, we mean a family of bounded operators $E(\cdot)$ defined on all Borel sets $\sigma$ of the plane, with the following properties.

(i) $E(\emptyset) = 0$, $E(\text{plane}) = I$, where $I$ is the identity.
(ii) For all $\sigma_1, \sigma_2$, $E(\sigma_1 \cap \sigma_2) = E(\sigma_1) \cdot E(\sigma_2)$ and for disjoint $\sigma_1, \sigma_2$

$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2)$.

(iii) There exists a constant $M$ such that

$\|E(\sigma)\| \leq M$

for all $\sigma$.

(iv) For $x \in \mathfrak{X}$ and $\{\sigma_n\}$ a sequence of disjoint Borel sets,

$E\left( \bigcup_{n=1}^{\infty} \sigma_n \right) x = \sum_{n=1}^{\infty} E(\sigma_n)x$.

If an operator $S$ admits a representation $S = \int \! dE(\sigma)$ where $E(\cdot)$ is a spectral measure then $S$ is a scalar operator. $T$ is a spectral operator if $T = S + N$ where $S$ is a scalar operator, $N$ is a quasi-nilpotent operator and $S$ commutes with $N$.

I. Lemma 1. If $T^n = I, n$ a positive integer, $I$ the identity on the Banach space $\mathfrak{X}$, then $T$ is a scalar operator, of the form $T = \sum_{i=1}^{n} w_i E_i$ where $E_i E_j = \delta_{ij} E_i$, $i, j = 1, \ldots, n$ and $\sum_{i=1}^{n} E_i = I$.

Proof. By Dunford's spectral mapping theorem the spectrum of $T$ can consist of at most the $n$th roots of unity. Then the resolvent $(T - zI)^{-1}$ is holomorphic in the rest of the plane, so by [3, p. 179], we have

$(1) \quad T = \sum_{i=1}^{n} N_i + \sum_{i=1}^{n} w_i E_i$

where the $N_i$s are quasi-nilpotent operators, the $E_i$s are idempotent.

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operators and the $w_i$s are the $n$th roots of unity. Also
\[ E_iE_j = 0, \quad i \neq j; \quad E_iN_i = N_iE_i = N_i; \quad E_iN_j = N_jE_i = 0, \quad i \neq j; \]
\[ E_iT = TE_i \quad \text{and} \quad \sum_{i=1}^{n} E_i = I. \]

Now we will show all $N_i$s are zero. Multiplying (1) by $E_i$ we obtain
\[ TE_i = (N_i + w_i I)E_i. \]
Raising this to the $n$th power we have $E_i = \sum_{l=0}^{n} \binom{n}{l} w^{n-l} N_i^l E_i$ or
\[ 0 = \sum_{l=1}^{n} \binom{n}{l} w^{n-l} N_i^l E_i. \]

Now for any element of the form $y = E_i x$ we have $\sum_{l=1}^{n} \binom{n}{l} w^{n-l} N_i^l y = 0$. This implies that $N_i^l y$ is linearly dependent on $\{y, N_i y, \ldots, N_i^{n-1} y\}$ and hence the linear subspace $M$, generated by this set is invariant under $N_i$. Since $N_i$ is quasi-nilpotent on $\mathfrak{A}$, $N_i^l|_M$ is nilpotent. Assume that $N_i^l|_M \neq 0$. Then there exists $u \in M$, $u = E_i v$ such that $N_i u \neq 0$, $N_i^2 u = 0$ because $N_i^l|_M$ is nilpotent. But then from (2) we have $N_i u = 0$ contrary to hypothesis. Hence $N_i = 0$ on $M$ and since $y$ was chosen arbitrarily from $E_i \mathfrak{A}$, we have $N_i E_i = 0$. Thus $N_i = N_i \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} N_i E_i = 0$. We may now write $T = \sum_{i=1}^{n} w_i E_i$ and from this conclude that $T$ is a scalar operator.

**Lemma 2.** Let $A$ and $B$ be commuting scalar operators on a Banach space $\mathfrak{A}$ where
\[ B = \sum_{i=1}^{k} a_i E_i \quad \text{and} \quad E_i E_j = 0, \quad i \neq j; \quad E_i^2 = E_i; \quad \sum_{i=1}^{k} E_i = I; \]
then $A \cdot B$ is a scalar operator.

**II. Theorem 1.** Let $T^n = S$, where $n$ is a positive integer and $S$ an invertible scalar operator on the Banach space $\mathfrak{A}$; then $T$ is a scalar operator.

**Proof.** Since $S$ is scalar, $S = \int z dF(z)$. Let $A = \int z^{1/n} dF(z)$ where $z^{1/n}$ is the principal $n$th root of $z$. Now $A$ is a scalar operator and $A^n = S$. Because $T$ commutes with $S$, $T$ commutes with the spectral measure $F(z)$ and hence $T$ commutes with $A$ (see [1, p. 329]). Let $TA^{-1} = B$, then $B^n = T^n A^{-n} = SS^{-1} = I$. By Lemma 1, $B$ is a scalar operator of the form $B = \sum_{i=1}^{n} w_i E_i$; and since $TA = AT$ we have $T = AB = BA$. We may therefore invoke Lemma 2 to conclude that $T$ is a scalar operator.

**Corollary 1.** Let $T^n = S$ where $n$ is a positive integer and $S$ is a
scalar operator on the Banach space $\mathcal{X}$ with zero an isolated point of the spectrum of $S$; then $T = S_1 + N$ where $S_1$ is scalar, $N$ is nilpotent, $S_1$ commutes with $N$ and $N^n = 0$. Thus $T$ is a spectral operator.

The proof is obtained by a slight modification in the proof of Theorem 1.

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**Theorem 2.** If $T^n = S$, $n$ a positive integer, $S$ an invertible spectral operator on a Banach space $\mathcal{X}$ then $T$ is a spectral operator.

**Proof.** Again the proof is a slight modification of previous methods. Let $S = A + N$, where $A$ is a scalar operator and $N$ a quasi-nilpotent operator which commutes with $A$. Then $[TA^{-1/n}]^n = S \cdot A^{-1} = (A + N)A^{-1} = I + NA^{-1}$. Now since $NA^{-1}$ is quasi-nilpotent, $\sigma(I + NA^{-1}) = 1$ so $\sigma(TA^{-1/n})$ can consist of at most the $n$th roots of unity. Now one can show by the argument of Lemma 1 that $TA^{-1/n} = \sum_{i=1}^{n} w_i E_i + Q$ where $Q$ is a quasi-nilpotent operator which commutes with all the idempotents $E_i$, $i = 1, \cdots, n$. Thus $T = A^{1/n} \sum_{i=1}^{n} w_i E_i + A^{1/n}Q$, where the first term on the right side is a scalar operator and the second is a quasi-nilpotent operator which commutes with it.

**Corollary 2.** If $T^n = S$, $n$ a positive integer, $S$ a spectral operator on a Banach space $\mathcal{X}$ where zero is an isolated point of the spectrum of $S$, then $T$ is a spectral operator.

The proof requires only a slight modification of the proof of Theorem 2.

**Example.** We now exhibit an example to show that Theorems 1 and 2 need not be true for operators which have zero as a limit point of the spectrum. Let $H = L_2[0, 1] \oplus L_2[0, 1]$ with the usual Hilbert space norm. For $[f_1, f_2] \in H$ define $T[f_1(s), f_2(s)] = [tf_1(s) + f_2(s), -sf_2(s)]$. Then elementary but tedious calculations with two X two matrices show that $T$ is neither a scalar nor a spectral operator. However $T^2[f_1(s), f_2(s)] = [s^2f_1(s), s^2f_2(s)]$ so $T^2$ is clearly a normal, thus a scalar operator.

**Bibliography**