ROOTS OF SCALAR OPERATORS

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Introduction. The existence of normal roots of normal operators is a well known consequence of the spectral theorem. Very simple examples show however that normal operators may have roots which are not normal (in fact the identity operator has non-normal roots). In this paper we will show that the invertible scalar operators on a Banach space possess only scalar operators as roots.

Preliminaries. The term operator will be used to mean a bounded linear transformation of a Banach space into itself. By a *spectral measure* on a Banach space $\mathfrak{X}$, we mean a family of bounded operators $E(\cdot)$ defined on all Borel sets σ of the plane, with the following properties.

(i) E (empty set) $=0$, E (plane) $=I$, where I is the identity.

(ii) For all σ_1, σ_2; $E(\sigma_1 \cap \sigma_2)=E(\sigma_1)\cdot E(\sigma_2)$ and for disjoint σ_1, σ_2

$$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2).$$

(iii) There exists a constant M such that

$$\|E(\sigma)\| \leqq M \qquad \text{for all } \sigma.$$

(iv) For $x \in \mathfrak{X}$ and $\{\sigma_n\}$ a sequence of disjoint Borel sets,

$$E\left(\bigcup_{n=1}^{\infty} \sigma_n\right)x = \sum_{n=1}^{\infty} E(\sigma_n)x.$$

If an operator S admits a representation $S=\int z dE(z)$ where $E(\cdot)$ is a spectral measure then S is a *scalar operator*. T is a *spectral operator* if $T=S+N$ where S is a scalar operator, N is a quasi-nilpotent operator and S commutes with N.

I. LEMMA 1. *If $T^n=I$, n a positive integer, I the identity on the Banach space $\mathfrak{X}$, then T is a scalar operator, of the form $T=\sum_{i=1}^n w_iE_i$ where $E_iE_j=\delta_{ij}E_i$, $i, j=1, \cdots, n$ and $\sum_{i=1}^n E_i=I$.*

PROOF. By Dunford's spectral mapping theorem the spectrum of T can consist of at most the nth roots of unity. Then the resolvent $(T-zI)^{-1}$ is holomorphic in the rest of the plane, so by [3, p. 179], we have

$$(1) \qquad T = \sum_{i=1}^{n} N_i + \sum_{i=1}^{n} w_iE_i$$

where the N_is are quasi-nilpotent operators, the E_is are idempotent

Received by the editors March 11, 1961 and, in revised form, August 23, 1961.

operators and the w_is are the nth roots of unity. Also

$$E_iE_j = 0, \quad i \neq j; \quad E_iN_i = N_iE_i = N_i; \quad E_iN_j = N_jE_i = 0, \quad i \neq j;$$

$$E_iT = TE_i \quad \text{and} \quad \sum_{i=1}^{n} E_i = I.$$

Now we will show all N_is are zero. Multiplying (1) by E_j we obtain $TE_j = (N_j + w_jI)E_j$. Raising this to the nth power we have $E_j = \sum_{i=0}^{n} \binom{n}{i} w^{n-i} N_j^i E_j$ or

$$0 = \sum_{i=1}^{n} \binom{n}{i} w^{n-i} N_j^i E_j. \tag{2}$$

Now for any element of the form $y = E_jx$ we have $\sum_{i=1}^{n} \binom{n}{i} w^{n-i} N_j^i y = 0$. This implies that $N_j^n y$ is linearly dependent on $\{y, N_j^1 y, \cdots, N_j^{n-1} y\}$ and hence the linear subspace M, generated by this set is invariant under N_j. Since N_j is quasi-nilpotent on $\mathfrak{X}$, $N_j|_M$ is nilpotent. Assume that $N_j|_M \neq 0$. Then there exists $u \in M$, $u = E_jv$ such that $N_ju \neq 0$, $N_j^2 u = 0$ because $N_j|_M$ is nilpotent. But then from (2) we have $N_ju = 0$ contrary to hypothesis. Hence $N_j \equiv 0$ on M and since y was chosen arbitrarily from $E_j\mathfrak{X}$, we have $N_jE_j = 0$. Thus $N_j = N_j \sum_{i=1}^{n} E_i = \sum_{i=1}^{n} N_jE_i = 0$. We may now write $T = \sum_{i=1}^{n} w_iE_i$ and from this conclude that T is a scalar operator. We will now quote Theorem 2 of [**2**, p. 452]:

LEMMA 2. *Let A and B be commuting scalar operators on a Banach space $\mathfrak{X}$ where*

$$B = \sum_{i=1}^{k} z_iE_i \quad \text{and} \quad E_iE_j = 0, \quad i \neq j; \qquad E_i \cdot E_i = E_i; \qquad \sum_{i=1}^{k} E_i = I;$$

then $A \cdot B$ is a scalar operator.

II. THEOREM 1. *Let $T^n = S$, where n is a positive integer and S an invertible scalar operator on the Banach space $\mathfrak{X}$; then T is a scalar operator.*

PROOF. Since S is scalar, $S = \int z dF(z)$. Let $A = \int z^{1/n} dF(z)$ where $z^{1/n}$ is the principal nth root of z. Now A is a scalar operator and $A^n = S$. Because T commutes with S, T commutes with the spectral measure $F(z)$ and hence T commutes with A (see [**1**, p. 329]). Let $TA^{-1} = B$, then $B^n = T^nA^{-n} = SS^{-1} = I$. By Lemma 1, B is a scalar operator of the form $B = \sum_{i=1}^{n} w_iE_i$; and since $TA = AT$ we have $T = AB = BA$. We may therefore invoke Lemma 2 to conclude that T is a scalar operator.

COROLLARY 1. *Let $T^n = S$ where n is a positive integer and S is a* *scalar operator on the Banach space $\mathfrak{X}$ with zero an isolated point of the spectrum of S; then $T=S_1+N$ where S_1 is scalar, N is nilpotent, S_1 commutes with N and $N^n=0$. Thus T is a spectral operator.*

The proof is obtained by a slight modification in the proof of Theorem 1.

The author would like to express his gratitude to Charles A. McCarthy for pointing out the following:

THEOREM 2. *If $T^n=S$, n a positive integer, S an invertible spectral operator on a Banach space $\mathfrak{X}$ then T is a spectral operator.*

PROOF. Again the proof is a slight modification of previous methods. Let $S=A+N$, where A is a scalar operator and N a quasi-nilpotent operator which commutes with A. Then $[TA^{-1/n}]^n=S\cdot A^{-1}$ $=(A+N)A^{-1}=I+NA^{-1}$. Now since NA^{-1} is quasi-nilpotent, $\sigma(I+NA^{-1})=1$ so $\sigma(TA^{-1/n})$ can consist of at most the nth roots of unity. Now one can show by the argument of Lemma 1 that $TA^{-1/n}$ $=\sum_{i=1}^{n} w_iE_i+Q$ where Q is a quasi-nilpotent operator which commutes with all the idempotents E_i, $i=1,\cdots,n$. Thus $T=A^{1/n}\sum_{i=1}^{n} w_iE_i+A^{1/n}Q$, where the first term on the right side is a scalar operator and the second is a quasi-nilpotent operator which commutes with it.

COROLLARY 2. *If $T^n=S$, n a positive integer, S a spectral operator on a Banach space $\mathfrak{X}$ where zero is an isolated point of the spectrum of S, then T is a spectral operator.*

The proof requires only a slight modification of the proof of Theorem 2.

EXAMPLE. We now exhibit an example to show that Theorems 1 and 2 need not be true for operators which have zero as a limit point of the spectrum. Let $H=L_2[0,1]\oplus L_2[0,1]$ with the usual Hilbert space norm. For $[f_1,f_2]\in H$ define $T[f_1(s),f_2(s)]$ $=[sf_1(s)+f_2(s), -sf_2(s)]$. Then elementary but tedious calculations with two$\times$two matrices show that T is neither a scalar nor a spectral operator. However $T^2[f_1(s),f_2(s)]=[s^2f_1(s), s^2f_2(s)]$ so T^2 is clearly a normal, thus a scalar operator.

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