A CHARACTERIZATION OF CERTAIN
QUASI-OPEN MAPPINGS

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1. Introduction. In this paper we shall deal with mappings (continuous functions) defined on regions (open and connected) subsets of a plane $P$ with the range of the functions also contained in $P$. In reference [1], M. K. Fort defined such a mapping $f$ to be minimal provided that for each closed 2-cell $N$ contained in the domain of $f$, $f(N) \subseteq g(N)$ for every mapping $g$ whose domain contains $N$ and which is such that $f| \text{Fr } N = g| \text{Fr } N$. The following result was obtained in [1].

*If $f$ is light, then $f$ is open if and only if $f$ is minimal.*

In this paper, we prove the following similar theorem for certain quasi-open mappings.

1.1. Theorem. Let $f: X \to P$ be a compact mapping defined on a simply connected region $X$ in a plane $P$ with $f(X) \subseteq P$. Then $f$ is quasi-open if and only if $f$ is minimal and for no $x \in f(X)$ is it true that $f^{-1}(x)$ separates $X$.

The sufficiency of the condition in 1.1 is established in §2 and the necessity in §3.

Recall that a mapping $f: X \to P$ is compact provided that for each compact set $K \subseteq f(X)$, $f^{-1}(K)$ is compact. $f$ is quasi-open provided that for any $y \in f(X)$ and any open set $U$ in $X$ containing a compact component of $f^{-1}(y)$, $y \in \text{int } f(U)$. Here and throughout the paper such terms as interior, closed, closure (cl), boundary (Fr) will always be relative to the containing plane.

With $f: X \to P$ a mapping into a plane $P$, if $R$ is an elementary region (a bounded plane region whose boundary consists of a finite number of simple closed curves) whose closure is in the domain of $f$ and $x \in P - f(\text{Fr } R)$, then there is defined the topological index or winding number $\mu(x, f, R)$. Use will be made of the following well-known properties of the index.

1.2. $\mu(x, f, R)$ is constant on every component of $P - f(\text{Fr } R)$.

1.3. Suppose $x \in P - f(\text{Fr } R)$. If $g: X \to P$ is such that $g(\bar{x}) = f(\bar{x})$ for $\bar{x} \in \text{Fr } (R)$, then $\mu(x, f, R) = \mu(x, g, R)$.

1.4. Let $R_j; j = 1, 2, \ldots, k$ be a sequence of pair-wise disjoint el-
mentary regions in $R$ such that

$$(\text{cl } R) \cdot f^{-1}(x) \subset \bigcup_{i} R_{i}.$$  

Then $\mu(x, f, R) = \sum_{i} \mu(x, f, R_{i}).$ (See Theorem 3 on p. 126 in [7].)

1.5. If $\mu(x, f, R) \neq 0$, then $x \in f(R)$.

1.6. If $N$ is a closed 2-cell and $f$ is a homeomorphism on $N$, then $\mu(x, f, \text{int } N) = \pm 1$ for each $x \in f(\text{int } N)$.

In terms of this index, Fort [1] proved the following.

1.7. With $f : X \rightarrow P$ as before, $f$ is minimal if and only if for each closed 2-cell $N$, $f(N) = f(\text{Fr } N) + \{ p \mid \mu(p, f, \text{int } N) \neq 0 \}$.

2. **Compact plane mappings.** In this section we make the following assumption.

2.1. $f : X \rightarrow P$ is a compact mapping from a simply connected region in a plane $P$ into $P$.

From a result of Whyburn (§10 in [4]), there exists a factorization $f = lm$ in which $m$ is monotone and compact and $l$ is light and compact. Throughout the middle space $m(X)$ will be designated as $M$.

If the further assumption is made that for no $x \in f(X)$ it is true that $f^{-1}(x)$ separates $X$, it can be shown that no component of $f^{-1}(x)$ separates $X$. But then for no $z \in M$, it is true that $m^{-1}(z)$ separates $X$. 2.3 below then follows from the following result of Whyburn. (See Note 1, p. 313 in [5].)

If $m$ is a monotone mapping from a plane $X$ onto a space $Y$, then $Y$ is a plane if and only if $m$ is compact and $m^{-1}(y)$ does not separate $X$ for each $y \in Y$.

2.3. Let $f$ be as in 2.1 and let $f = lm$ be a factorization as before. In addition assume that for $x \in f(X)$, $f^{-1}(x)$ does not separate $X$. Then the middle space $M$ is a topological plane.

2.4. Let $f : X \rightarrow P$, $l$, $m$, $M$ be as in 2.3. Then if $N$ is a closed 2-cell such that $N \subset M$, $m^{-1}(\text{int } N)$ is a simply connected region.

**Proof.** $M - \text{int } N$ is connected. Therefore, since $m$ is compact and monotone, $m^{-1}(M - \text{int } N) = X - m^{-1}(\text{int } N)$ is connected. (See 8.3 in [4].) Since $m$ is compact, $m^{-1}(N)$ and thus $\text{cl } (m^{-1}(\text{int } N))$ is compact. Thus $m^{-1}(\text{int } N)$ is a plane region whose closure is compact and whose complement in the plane is connected. Hence $m^{-1}(\text{int } N)$ is simply connected.

2.5. Let $f : X \rightarrow P$ be a compact mapping. Let $x \in f(X)$ and let $U$ be an open set such that $f^{-1}(x) \subset U$. Then there exists an open set $V \subset P$ such that $x \in V$ and $f^{-1}(V) \subset U$.

This follows easily from the compactness of $f$.

2.6. Let $f : X \rightarrow P$ be a mapping as in 2.3. Furthermore assume that $f$
is minimal. Then \( f \) is quasi-open.

**Proof.** Let \( f = lm \) be a factorization as in 2.3. By 2.3 the middle space \( M \) may be taken to be a complex plane. Let \( x \in f(X) \) and suppose \( K \) is a component of \( f^{-1}(x) \). Let \( U \) be an open set in \( X \) such that \( K \subset U \). From the definition of quasi-open, the proof will be complete if we show that \( x \) is an interior point of \( f(U) \). Toward this end, let \( p = m(K) \). Note that \( p \in l^{-1}(x) \). Since \( l^{-1}(x) \) is totally disconnected it can be shown that there exists arbitrarily small 2-cells \( N \) such that \((\text{Fr } N) \cdot l^{-1}(x) = \emptyset \) and \( p \in \text{int } N \). By 2.5, \( N \) may be chosen so that \( m^{-1}(N) \subset U \). Let \( G = m^{-1}(\text{int } N) \). By 2.4, \( G \) is simply connected. Since \( f \) is compact, \( f^{-1}(x) \cdot \text{cl } G \) is compact and since \( f^{-1}(x) \cdot \text{Fr } G = \emptyset \) it follows that \( f^{-1}(x) \cdot G \) is compact. Then since \( G \) is simply connected, there exists a closed 2-cell \( T \) such that \( f^{-1}(x) \cdot G \subset \text{int } T \subset T \subset G \). Further by 1.7 \( \mu(x, f, \text{int } T) \neq 0 \). But then from 1.2 and 1.5, \( f(T) \supset Q_x \) where \( Q_x \) is the component of \( P - f(\text{Fr } T) \) that contains \( x \). Since \( Q_x \) is open and \( x \in Q_x \subset f(T) \subset f(U) \) it follows that \( x \) is an interior point of \( f(U) \).

3. **Compact quasi-open mappings.** In this section we make the following assumption.

3.1. \( f : X \to P \) is a compact and quasi-open mapping defined on a simply connected region \( X \) in the plane \( P \) and \( f(X) \subset P \).

Assuming 3.1, it is known (10.2 and 10.4 in [4]) that there is a monotone-light open factorization \( f = lm \) for which \( m \) is a compact monotone mapping and \( l \) is compact, light, and open relative to \( f(X) \). Further from the definition of quasi-open as given in the introduction, \( f(X) \) is open in \( P \) and hence \( l \) is open relative to \( P \). Furthermore, we have the following.

3.2. For no point \( x \in f(X) \) is it true that \( f^{-1}(x) \) separates \( X \).

**Proof.** Suppose \( f^{-1}(x) \) separates \( X \). Then there exists a separation \( X - f^{-1}(x) = A + B \).

Since \( f^{-1}(x) \) is compact \( A + f^{-1}(x) \) or \( B + f^{-1}(x) \) is compact. Assume that \( A + f^{-1}(x) \) is compact. Then \( f(A) + x \) is compact. However \( A \) is open and is the union of components of point inverses. Hence from the quasi-openness of \( f \), \( f(A) \) is open. This is a contradiction since then \( x \) would be the boundary of an open set \( f(A) \) in the plane.

On the basis of 3.2 and 2.3 with \( f \) as in 3.1, the middle space \( M \) can be taken to be a plane. Recall that \( f(X) \) is open in \( P \). Thus there is available the following result of Whyburn.

3.3. (See VIII, 1.1 and 1.11 in [3].) There exists an integer \( k \) and a completely scattered set \( D \subset f(X) \), such that for each \( x \in f(X) - D \), \( l^{-1}(x) \) consists of \( k \) distinct points. Furthermore on the set \( M - l^{-1}(D) \), \( l \) is a
local homeomorphism. [Note that for each point $x \in D$, $x$ is an isolated point of $D$. Also $D$ is closed.]

By making use of 3.3 and 2.4, the following can be obtained.

3.4. For each $z \in f(X) - D$, there exists arbitrarily small closed 2-cells $N$ such that $z \in \text{int } N \subset N \subset f(X) - D$ and such that $f^{-1}($int $N)$ is the union of a finite number of pairwise disjoint open two-cells $Q_i$ for which $f(Q_i) = $int $N$ and $f^{-1}(z*) \cdot Q_i$ consists of a single component of $f^{-1}(z*)$ for each $z* \in $int $N$.

3.5. $X - f^{-1}(D)$ is a connected open set.

Proof. Notice that $X - f^{-1}(D) = m^{-1}(M - l^{-1}(D))$ is open and furthermore is connected since $M - l^{-1}(D)$ is connected and $m$ is a compact monotone mapping.

3.6. Definition of $\mu(x)$. On the open connected set $S = X - f^{-1}(D)$ define the function $\mu$ as follows. Let $x \in S$. Then $f^{-1}f(x)$ consists of exactly $k$ compact components. (See 3.3.) Let $K_x$ be that component that contains $x$. There exists an elementary region $R$ such that

$$K_x \subset R \subset \text{cl } R \subset X - f^{-1}(D)$$

and

$$f^{-1}(x) \cdot \text{cl } R = K_x.$$

It will be shown in 3.7 that $\mu(x, f, R)$ is independent of the particular choice of $R$ provided that $R$ satisfies (1). On that basis the following may be defined:

$$\mu(x) = \mu(f(x), f, R).$$

(Because of the nature of point inverses of $f$, $\mu$ is a point function version of the function defined in II. 3.4 of [7].)

3.7. $\mu(x, f, R_1) = \mu(x, f, R_2)$ provided that $R_1$ and $R_2$ satisfy (1).

Proof. Let $Q$ be the component of $R_1 \cdot R_2$ that contains $K_x$. There exists an elementary region $R$ such that

$$K_x \subset R \subset \text{cl } R \subset Q \subset R_1 \cdot R_2.$$

From 1.4, $\mu(f(x), f, R_1) = \mu(f(x), f, R_2) = \mu(f(x), f, R_2)$.

Use will be made of the following in investigating the function $\mu$.

3.8. Young's Modification Theorem. (See [6].) Let $N$ be a closed 2-cell with interior $G$ and boundary $J$. Suppose $X$ is a continuum and $m : X \rightarrow N$ is a monotone mapping of $X$ onto $N$ such that $m^{-1}(G)$ is an open 2-cell. Then there is a mapping $h : X \rightarrow N$ of $X$ onto $N$ such that $h(x) = m(x)$ for $x \in m^{-1}(J)$ and $h^{-1}h(x) = x$ for $x \in m^{-1}(G)$.

3.9. $|\mu(x)| = 1$ for each $x \in X - f^{-1}(D)$.

Proof. Let $x \in X - f^{-1}(D)$. Let $R$ be an elementary region admissible for the computation of $\mu(x)$. (See 3.6.) Let $y = m(x) \in M$. Note that $l$ is a local homeomorphism at $y$ and recall that $m$ is compact.
Hence by 2.5 there exists a closed 2-cell neighborhood \( N \) of \( y \) such that \( m^{-1}(N) \subset R \) and such that \( I \) is a homeomorphism on \( N \). Let \( G = m^{-1}(\text{int } N) \). By 2.4 \( G \) is an open 2-cell. Thus 3.8 can be applied to \( m \mid m^{-1}(N) \) and using this a modification \( h : X \to M \) of \( m \) can be defined such that \( h(z) = m(z) \) for \( z \in X - G \) and \( h \) is one-to-one on \( G \). Let \( z^* \) be the unique point in \( G \) such that \( h(z^*) = y \). Note also \( h^{-1}(y) \cdot R = z^* \). Let \( W \) be a closed 2-cell such that \( z^* \in \text{int } W \subset \text{int } G \). Note that \( \partial h \mid W \) is a homeomorphism. Then by 1.6 and from the fact that \( \partial h(z^*) = \partial h(y) = f(x) \), it follows that \( \mu(f(x), \partial h, \text{int } W) = \pm 1 \). Further since \( \partial ((\partial h)^{-1}f(x)) \cdot R = z^* \), it follows from 1.4 that \( \mu(f(x), \partial h, \text{int } W) = \mu(f(x), \partial h, R) \). Moreover by 1.3, since \( f = \partial h \mid \text{Fr}(R) \), \( \mu(f(x), f, R) = \mu(f(x), \partial h, R) \). Combining this with the two previous equalities involving \( \mu \) yields \( \mu(f(x), f, R) = \pm 1 \).

3.10. \( \mu \) is continuous on \( X - f^{-1}(D) \). (Hence by 3.5 \( \mu(x) \) is constant on \( X - f^{-1}(D) \).)

**Proof.** Let \( x \in X - f^{-1}(D) \) and \( K_x \) the component of \( f^{-1}(x) \) that contains \( x \). Choose \( R \) as in (1) of 3.6 so that \( \mu(x) = \mu(f(x), f, R) \). Choose a closed 2-cell \( N \) such that \( f(x) \in \text{int } N \subset \text{cl } N \subset f(X) - f(\text{Fr}(R)) \) and is chosen as in 3.4. Let \( Q \) be the component of \( f^{-1}(\text{int } N) \) that contains \( K_x \). Since \( f(Q_i) = \text{int } N \) for each component \( Q_i \) of \( f^{-1}(\text{int } N) \) (see 3.4) and since \( f^{-1}(x) \cdot R = K_x \), it follows that \( Q \) is the only component of \( f^{-1}(\text{int } N) \) that intersects \( R \).

Now let \( y \in Q \). Let \( K_y \) be the component of \( f^{-1}(f(y)) \) that contains \( y \). However from the way in which \( N \) was chosen (see 3.4) \( K_y \) is the only component of \( f^{-1}(f(y)) \) that intersects \( R \). Thus \( R \) is admissible for the computation of \( \mu(y) \) and \( \mu(y) = \mu(f(y), f, R) \). However by 1.2, \( \mu(f(y), f, R) = \mu(f(x), f, R) \) and thus \( \mu(x) = \mu(y) \).

3.11. **Let** \( f : X \to P \) **be as in 3.1. Then** \( f \) **is minimal.**

**Proof.** Let \( N \) be a closed 2-cell in \( X \). From 1.5 and 1.7 we need show only that \( \mu(x, f, \text{int } N) \neq 0 \) for each \( x \in f(N) - f(\text{Fr } N) \). Let \( x \in f(N) - f(\text{Fr } N) \). Define \( D \) as in 3.3. Let \( U_x \) be the component of \( P - f(\text{Fr } N) \) that contains \( x \). There exists a point \( y \in U_x \cdot [f(\text{int } N) - D] \). This follows from the quasi-openness of \( f \) and the definition of \( D \). By 1.2, \( \mu(y, f, \text{int } N) = \mu(x, f, \text{int } N) \). Let \( K_1, \ldots, K_a \) be the components of \( f^{-1}(y) \) that intersect and hence are contained in \( \text{int } N \). There exist elementary regions \( R_1, R_2, \ldots, R_a \) such \( K_i \subset R_i \subset \text{cl } R_i \subset \text{int } N \) and \( R_i \cdot R_j = \emptyset \) if \( i \neq j \). By 1.4, \( \mu(y, f, \text{int } N) = \sum_i \mu(y, f, R_i) \). Let \( x_i \in K_i \). Thus, from definition of \( \mu \), 3.9, and 3.10 \( | \sum_i \mu(x, f, R_i) | = | \sum_i \mu(x, f, R_i) | = h \) and \( \mu(x, f, \text{int } N) = \mu(y, f, \text{int } N) \neq 0 \).

**Theorem 1.1.** The validity of the Theorem 1.1 stated in the introduction now follows from 2.6, 3.2 and 3.11.
References


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