THE OPERATION OF THE UNIVERSAL
DOMAIN ON THE PLANE1

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The problem we are considering is if

$$\forall (t, x, y) \rightarrow (P(t, x, y), Q(t, x, y))$$

defines a regular operation of the additive group of the universal
domain $\Omega$ on the affine plane $A^2$, then what can be said about the form
of the polynomials $P$ and $Q$?

First we note that any change of coordinates for $A^2$ is defined by
equations

$$u = M(x, y), \quad x = R(u, v),$$
$$v = N(x, y), \quad y = S(u, v),$$

where $M, N, R, S$ are polynomials with coefficients in $\Omega$. Now the
operation of $\Omega$ on $A^2$ given by equations (1) is defined in $uv$-coordinates by

$$\forall (t, u, v) \rightarrow (M(P(t, R, S), Q(t, R, S)), N(P(t, R, S), Q(t, R, S))).$$

We shall use a theorem from a paper by Engel [1] which states that
in a change of coordinates such as (2) the degree of $M$ must divide the
degree of $N$ or vice versa. Engel's proof is for characteristic $= 0$.
Assuming this result we will prove the following theorem.

**Theorem.** If the characteristic of $\Omega$ is 0 and if there is a regular
operation of $\Omega$ on $A^2$, then there is a change of coordinates of $A^2$ such
that in terms of the $uv$-coordinates the given operation has the form

$$(t, u, v) \rightarrow (u, v + tf(u)) \quad \text{with } f \in \Omega[u].$$

We will use the theory of algebraic groups as developed by Rosen-
licht in [2]. If $H \in \Omega(x, y)$ and $t \in \Omega$, then $\lambda H$ is the function
$H(P(t, x, y), Q(t, x, y))$. $H$ is called *invariant* if $\lambda H = H$ for all $t \in \Omega$.
Now to prove the theorem we proceed by proving a sequence of
three lemmas.

**Lemma 1.** There exists a nonconstant polynomial in $\Omega[x, y]$ which is
invariant.

**Proof.** Let $k$ be an algebraically closed field of definition for $\Omega, A^2$,
and the operation on $A^2$. Consider the variety $W$ of $\Omega$-orbits on $A^2$. If this were of dimension 0, any two independent generic points for $A^2$ over $k$ would belong to the same orbit. This is impossible and therefore $\dim(W) > 0$.

Now $\dim(W) > 0$ implies that the field $k(W)$ has a nonconstant function. But $k(W)$ is $k$-isomorphic to the subfield of invariant functions in $k(x, y)$. Hence there exists a nonconstant function $H$ in $k(x, y)$ which is invariant.

Let $H = H_1/H_2$ where $H_1 \subseteq k[x, y]$ and $H_1$ and $H_2$ are relatively prime. Let $t$ be a variable over $k(x, y)$. Now $\lambda_i H = H$ implies $\lambda_i H_1/\lambda_i H_2 = H_1/H_2$. Using the unique factorization of $k[t, x, y]$ it can be shown that $\lambda_i H_i = H_i$. Since $H$ is a nonconstant function, either $H_1$ or $H_2$ is the required nonconstant invariant polynomial in $k[x, y]$. Q.e.d.

Note that by taking powers of such an invariant polynomial we can obtain an invariant polynomial of arbitrarily high degree.

For the operation defined by polynomials (1) we must have $P(0, x, y) = x$ and $Q(0, x, y) = y$. Therefore these polynomials must be of the form

$$P(t, x, y) = x + tf_1(x, y) + t^2f_2(x, y) + \cdots + t^nf_n(x, y),$$
$$Q(t, x, y) = y + tg_1(x, y) + t^2g_2(x, y) + \cdots + t^mg_m(x, y).$$

Lemma 2. If the polynomials $P$ and $Q$ which define the operation of $\Omega$ on $A^2$ are such that $f_n \neq 0$, $g_m \neq 0$, $0 < n \leq m$, and $m/n = l/j$ with $(l, j) = 1$, then $f_n = c g_m$ where $c$ is a constant.

Proof. Let the weight of a monomial $x^\nu y^\mu$ be defined as the integer $\nu + \mu$. The weight of a polynomial will be the maximum of the weights of its monomials. Now

$$\lambda_i (x^\nu y^\mu) = P_i Q^\mu = (t^n g_m)^{\nu + \mu} + \text{terms of lower degree in } t.$$ 

By Lemma 1 there is a nonconstant polynomial $H(x, y)$ such that $\lambda_i H = H$. If $s$ is the weight of $H$ and if $x^\nu y^\mu$ is the monomial of $H$ of weight $s$ and with degree in $x$ maximal, then the part of $H$ of weight $s$ must be of the form

$$H_s(x, y) = a_0 x^\nu y^\mu + a_1 x^{\nu-1} y^{\mu+1} + a_2 x^{\nu-2} y^{\mu+2} + \cdots + a_s x^{\nu-s} y^{\mu+s}.$$ 

The coefficient of $t^l$ in $\lambda_i H$ will be $H_s(f_n, g_m)$ which must be identically zero since $\lambda_i H = H$. From this we obtain a relation of the form

$$af_n^{q1} + af_n^{(q-1)l} g_m + \cdots + a_q g_m = 0.$$ 

This implies that $f_n^{q1}/g_m^l$ must equal one of the roots of an algebraic
equation of degree q. Therefore \( f_n' = cg_m' \) where \( c \) is some constant.

Q.e.d.

**Lemma 3.** If the polynomials \( P \) and \( Q \) in (4) are such that \( f_n \neq 0, g_m \neq 0, 0 < n \leq m \), then \( n|m \).

**Proof.** Choose an invariant polynomial \( H(x, y) \) so that the degree of \( H \) is greater than the maximum of the degrees of

\[
\{f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m\}.
\]

Let \( v \) be the degree of \( H \).

Lemma 2 implies that if \( f_n' \) and \( g_m' \) are nonconstant polynomials, then \( g_m = G^l(x, y) \) and \( f_n = cG^l(x, y) \) where \( c \) is a constant and \( l \) and \( j \) are as in Lemma 2. Let \( \mu \) be the degree of \( G \). (If \( f_n \) and \( g_m \) are constants, \( \mu = 0 \).)

Now since \( (t, x, y) \rightarrow (P(t, x, y), Q(t, x, y)) \) defines a regular operation of \( \Omega \) on \( A^2 \), it follows that

\[
\Omega[l, x, y] = \Omega[l, P(l, x, y), Q(l, x, y)].
\]

If we let \( t = H(x, y) \) in (5) we have that

\[
\Omega[x, y] = \Omega[H, x, y] = \Omega[H, P(H, x, y), Q(H, x, y)].
\]

Since \( H \) is invariant \( H(x, y) = H(P(H, x, y), Q(H, x, y)) \). Thus (6) implies

\[
\Omega[x, y] = \Omega[P(H, x, y), Q(H, x, y)].
\]

From the choice of \( v \) it follows that the degree of the first generator on the right side of (7) must be the degree of \( H^* f_n \) which is \( nm + j \mu \).

Similarly the degree of the second generator on the right side of (7) is the degree of \( H^* g_m \) which is \( mv + l \mu \).

By the result in [1] that we referred to above, \( (mv + j \mu) \) divides \( (mv + l \mu) \). This implies that \( j|l \). But since \( (j, l) = 1, j = 1 \) and \( n|m \). Q.e.d.

We will now prove the theorem. Suppose we have polynomials \( P \) and \( Q \) of form (4) with \( n \neq 0 \) and \( m \neq 0 \). Let \( n \leq m \). By Lemma 3 we know that \( n|m \) so that in the notation of Lemma 2, \( m/n = l \) and \( j = 1 \). Thus Lemma 2 implies that \( g_m = cf^l_n \) where \( c \) is a constant.

Make the following change of coordinates for \( A^2 \):

\[
u = x, \quad x = u,
\]

\[
v = y - cx^l, \quad y = v + cu^l.
\]

Applying equation (3) to this particular change of coordinates, the
given operation will be defined in $uv$-coordinates by

$$(8) \quad (i, u, v) \rightarrow (P(i, u, v + cu'), Q(i, u, v + cu')).$$

The coefficient of $t^m$ in $Q(i, u, v + cu') - cP'(i, u, v + cu')$ is $g_{m}(u, v + cu') - cf_{m}(u, v + cu')$, which is zero.

Hence (8) reduces to an expression of form (4) in which the degree of $P$ in $t$ is $n$ while the degree of $Q$ in $t$ is less than $m$. We can repeat this process until we obtain coordinates in which either $P$ or $Q$ will have degree 0 in $t$.

Thus for any regular operation of $\Omega$ on $\mathbb{A}^2$, there are coordinates $u, v$ so that the operation has the form

$$(t, u, v) \rightarrow (u, Q(t, u, v)).$$

But we may consider $(t, v) \rightarrow (Q(t, u, v))$ as defining an operation on the line. Now $Q$ must satisfy the identity

$$Q(s + t, u, v) = Q(s, u, Q(t, u, v)).$$

Since characteristic = 0, it follows that $Q(t, u, v)$ must be of the form $v + tf(u)$ where $f \in \Omega[u]$. Thus in $uv$-coordinates the given operation has the form

$$(t, u, v) \rightarrow (u, v + tf(u)).$$

This completes the proof of the theorem.

REFERENCES


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