CONCERNING NONNEGATIVE VALUED INTERVAL FUNCTIONS

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1. **Introduction.** In this paper we prove that if H is a real nonnegative valued function of subintervals of the number interval $[a, b]$, then the integral (§2)

$$\int_{[a,b]} H(I)$$

exists if and only if for each real valued nondecreasing function m on $[a, b]$ the integral

$$\int_{[a,b]} [H(I)dm]^{1/2}$$

exists.

2. **Preliminary definitions and theorems.** Throughout this paper all integrals considered will be Hellinger [1] type limits of the appropriate sums (the definitions, theorems and proofs of this paper can be extended to "many valued" functions). Thus, if K is a real valued function of subintervals of the number interval $[a, b]$, the existence of $\int_{[a,b]}K(I)$ is necessary and sufficient for the existence of $\int_I K(J)$ for each subinterval I of $[a, b]$. Furthermore, the function F of subintervals of $[a, b]$, defined by $F(I) = \int_I K(J)$, is additive.

If K is a real valued function of subintervals of $[a, b]$, then the statement that K is Σ-bounded on $[a, b]$ means that there is a subdivision D of $[a, b]$ such that the set of sums $\sum_E K(I)$, where E is a refinement of D and the sum is taken over all intervals I of E, is bounded. This implies that if I is an interval in a refinement of D, then the set of sums $\sum_Q K(J)$, where Q is a subdivision of I and the sum is taken over all intervals J of Q, has a least upper bound $L(I)$ and a greatest lower bound $G(I)$. We now see that if each of R and T is a refinement of D, and S is a refinement of each of R and T, then $\sum_R G(I) \leqq \sum_S G(I) \leqq \sum_S K(I) \leqq \sum_S L(I) \leqq \sum_T L(I)$. From this it follows that each of $\int_{[a,b]}G(I)$ and $\int_{[a,b]}L(I)$ exists, that $\int_{[a,b]}G(I) \leqq \int_{[a,b]}L(I)$, and that $\int_{[a,b]}K(I)$ exists if and only if $\int_{[a,b]}G(I) = \int_{[a,b]}L(I)$.

We state without proof a theorem of Kolmogoroff [2].

THEOREM. *If K is a real valued function of subintervals of $[a, b]$,*

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integrable on $[a, b]$, *then* $\int_{[a,b]} |K(I) - \int_I K(J)| = 0$.

From this we have the following theorem.

THEOREM 1. *If K is a real valued function of subintervals of $[a, b]$ integrable on $[a, b]$ and c is a positive number, then there is a subdivision E of $[a, b]$ such that if F is a refinement of E and, for each interval I of F, $M(I)$ is either $K(I)$ or $\int_I K(J)$, then $| \sum_F M(I) - \int_{[a,b]} K(I)| < c$.*

PROOF. By Kolmogoroff's theorem there is a subdivision E of $[a, b]$ such that if F is a refinement of E, then $\sum_F |K(I) - \int_I K(J)| < c$, so that if for each I in F, $M(I)$ is either $K(I)$ or $\int_I K(J)$, then

$$\left| \sum_F M(I) - \int_{[a,b]} K(I) \right| \leq \sum_F \left| M(I) - \int_I K(J) \right|$$
$$\leq \sum_F \left| K(I) - \int_I K(J) \right| < c.$$

3. Two theorems about real nonnegative valued interval functions. We now use Theorem 1 to prove an elementary theorem about an integral of the form

$$\int_{[a,b]} \left[K(I) \int_I K(J) \right]^{1/2}.$$

THEOREM 2. *If K is a real nonnegative valued function of subintervals of $[a, b]$, integrable on $[a, b]$, then*

$$\int_{[a,b]} \left[K(I) \int_I K(J) \right]^{1/2} = \int_{[a,b]} K(I).$$

PROOF. If I is a subinterval of $[a,b]$, then either $K(I) \leq [K(I)\int_I K(J)]^{1/2} \leq \int_I K(J)$ or $\int_I K(J) \leq [K(I)\int_I K(J)]^{1/2} \leq K(I)$, so that for each subdivision D of $[a, b]$ there is a sum $\sum_D N(I)$ and a sum $\sum_D M(I)$ such that for each I in D, each of $N(I)$ and $M(I)$ is either $K(I)$ or $\int_I K(J)$ and $\sum_D N(I) \leq \sum_D [K(I)\int_I K(J)]^{1/2} \leq \sum_D M(I)$. Therefore, by Theorem 1, $\int_{[a,b]} [K(I)\int_I K(J)]^{1/2}$ exists and is $\int_{[a,b]} K(I)$.

THEOREM 3. *If each of H and K is a real nonnegative valued function of subintervals of $[a, b]$, integrable on $[a, b]$, then*

$$\int_{[a,b]} [H(I)K(I)]^{1/2} = \int_{[a,b]} \left[\left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right]^{1/2}$$

PROOF. We first see that if each of A, B, C, and D is a nonnegative number, then

$$[(A+B)(C+D)]^{1/2} = [(\{A^{1/2}\}^2 + \{B^{1/2}\}^2)(\{C^{1/2}\}^2 + \{D^{1/2}\}^2)]^{1/2} \geqq (AC)^{1/2} + (BD)^{1/2}.$$

This implies that if R is a refinement of the subdivision S of $[a, b]$, then $\sum_R [\{\int_I H(J)\}\{\int_I K(J)\}]^{1/2} \leqq \sum_S [\{\int_I H(J)\}\{\int_I K(J)\}]^{1/2}$, so that the greatest lower bound of all sums $\sum_T [\{\int_I H(J)\}\{\int_I K(J)\}]^{1/2}$ for all subdivisions T of $[a, b]$, is $\int_{[a,b]} [\{\int_I H(J)\}\{\int_I K(J)\}]^{1/2}$. If E is a subdivision of $[a, b]$, then

$$\begin{aligned}
&\left| \sum_E [H(I)K(I)]^{1/2} - \sum_E \left[\left\{ \int_I H(J) \right\} \left\{ \int_I K(J) \right\} \right]^{1/2} \right| \\
&= \left| \sum_E [H(I)]^{1/2} \left[\{K(I)\}^{1/2} - \left\{ \int_I K(J) \right\}^{1/2} \right] \right. \\
&\qquad \left. + \sum_E \left[\{H(I)\}^{1/2} - \left\{ \int_I H(J) \right\}^{1/2} \right] \left[\int_I K(J) \right]^{1/2} \right| \\
&\leqq \left\{ \sum_E H(I) \right\}^{1/2} \left\{ \sum_E K(I) - 2\left[K(I) \int_I K(J) \right]^{1/2} + \int_I K(J) \right\}^{1/2} \\
&\quad + \left\{ \sum_E H(I) - 2\left[H(I) \int_I H(J) \right]^{1/2} + \int_I H(J) \right\}^{1/2} \\
&\quad \cdot \left\{ \sum_E \int_I K(J) \right\}^{1/2},
\end{aligned}$$

so that by Theorem 2, $\int_{[a,b]} [H(I)K(I)]^{1/2}$ exists and is $\int_{[a,b]} [\{\int_I H(J)\}\{\int_I K(J)\}]^{1/2}$.

4. The existence theorem. We now establish the main result of this paper.

THEOREM 4. *If H is a real nonnegative valued function of subintervals of $[a, b]$, then the following two statements are equivalent:*

(1) *If m is a real valued nondecreasing function on $[a, b]$, then $\int_{[a,b]} [H(I)dm]^{1/2}$ exists, and*

(2) *$\int_{[a,b]} H(I)$ exists.*

PROOF. The fact that (1) follows from (2) is an immediate consequence of Theorem 3.

Suppose (1) is true. We first show that if x is in $[a, b]$, then there is a positive number d and a number M such that if y is in $[a, b]$ and $x-d<y<x$, then $H[y, x] \leqq M$, and if $x<y<x+d$, then $H[x, y] \leqq M$. For suppose $a<y\leqq b$ and there is an increasing sequence $z_1, z_2, z_3, \cdots$ of numbers of $[a, b]$ such that $0<H[z_n, y] \leqq H[z_{n+1}, y]$ for each positive integer n, and $z_n \to y$ and $H[z_n, y] \to \infty$ as $n \to \infty$. There is a non-

decreasing function m on $[a, b]$ such that $m(y)-m(z_n)=H[z_n, y]^{-1/2}$ for each positive integer n, so that $[H[z_n, y]\{m(y)-m(z_n)\}]^{1/2}\rightarrow\infty$ as $n\rightarrow\infty$. This implies that $\int_{[a,b]}[H(I)dm]^{1/2}$ does not exist, a contradiction. A similar argument holds for $[y, b]$ if $a\leqq y<b$.

We now show that H is Σ-bounded on $[a, b]$. Suppose that this is not true. The greatest lower bound t of the set of all nunbers x, such that $a<x\leqq b$ and H is not Σ-bounded on $[a, x]$, is such that either (1) $a<t\leqq b$ and if $a\leqq y<t$, then H is not Σ-bounded on $[y, t]$; or (2) $a\leqq t<b$ and if $t<y\leqq b$, then H is not Σ-bounded on $[t, y]$. We assume the former of these cases. A contradiction follows from the latter in a similar manner. This and the preceding paragraph imply that there is an increasing sequence $z_1, z_2, z_3, \cdots$ of numbers of $[a, b]$ such that $z_n\rightarrow t$ and $\sum_{k=1}^n H[z_k, z_{k+1}]\rightarrow\infty$ as $n\rightarrow\infty$. It follows that there is a sequence $b_1, b_2, b_3, \cdots$ of nonnegative numbers such that $\sum_{k=1}^n b_k$ converges and $\sum_{k=1}^n (H[z_k, z_{k+1}]b_k)^{1/2}\rightarrow\infty$ as $n\rightarrow\infty$. There is a nondecreasing function m on $[a, b]$ such that for each positive integer n, $m(z_{n+1})-m(z_n)=b_n$. Since $z_n\rightarrow t$ and $\sum_{k=1}^n (H[z_k, z_{k+1}]\{m(z_{k+1})-m(z_k)\})^{1/2}\rightarrow\infty$ as $n\rightarrow\infty$, it follows that $\int_{[a,b]}[H(I)dm]^{1/2}$ does not exist, a contradiction.

Therefore there is a subdivision D of $[a, b]$ and a number M such that if E is a refinement of D, then $\sum_E H(I)\leqq M$. For each interval I of a refinement of D, the set of sums $\sum_Q K(J)$, where Q is a subdivision of I, has a least upper bound $L(I)$ and a greatest lower bound $G(I)$. We let each of l and g denote a function on $[a, b]$ such that $l(a)=g(a)=0$, and for $a<x\leqq b$, $l(x)=\int_{[a,x]}L(J)$ and $g(x)=\int_{[a,x]}G(J)$. We see that each of l and g is a real valued nondecreasing function on $[a, b]$.

Suppose m is a nondecreasing function on $[a, b]$ and c is a positive number. There is a subdivision A of $[a, b]$ such that if each of E and E' is a refinement of A, then $\left|\sum_E [H(I)\Delta m]^{1/2}-\sum_{E'} [H(I)\Delta m]^{1/2}\right| <c/16$. By Theorem 3 there is a refinement B of D such that if each of F and F' is a refinement of B, then

$$\left|\int_{[a,b]}[dldm]^{1/2}-\sum_F [L(I)\Delta m]^{1/2}\right|<c/8$$

and

$$\left|\int_{[a,b]}[dgdm]^{1/2}-\sum_{F'} [G(I)\Delta m]^{1/2}\right|<c/8.$$

There is a common refinement D' of A and B. For each interval I in D' there is a subdivision R_I and a subdivision S_I of I such that $0\leqq L(I)-\sum_{R_I} H(J)<c^2/16N$ and $0\leqq\sum_{S_I} H(J)-G(I)<c^2/16N$, where N is the number of intervals in D', so that

$$0 \leqq \sum_{R_I} \{L(J) - H(J)\} < c^2/16N$$

and

$$0 \leqq \sum_{S_I} \{H(J) - G(J)\} < c^2/16N.$$

It follows that

$$\begin{aligned} 0 &\leqq \sum_{D'} \sum_{R_I} \{[L(J)]^{1/2} - [H(J)]^{1/2}\}\{\Delta m\}^{1/2} \\ &\leqq \left\{ \sum_{D'} \sum_{R_I} L(J) - 2[L(J)H(J)]^{1/2} + H(J) \right\}^{1/2} \left\{ \sum_{D'} \sum_{R_I} \Delta m \right\}^{1/2} \\ &\leqq \left\{ \sum_{D'} \sum_{R_I} L(J) - 2H(J) + H(J) \right\}^{1/2} \{m(b) - m(a)\}^{1/2} \\ &\leqq (c/4)\{m(b) - m(a)\}^{1/2}, \end{aligned}$$

and

$$\begin{aligned} 0 &\leqq \sum_{D'} \sum_{S_I} \{[H(J)]^{1/2} - [G(J)]^{1/2}\}\{\Delta m\}^{1/2} \\ &\leqq \left\{ \sum_{D'} \sum_{S_I} H(J) - 2[H(J)G(J)]^{1/2} + G(J) \right\}^{1/2} \left\{ \sum_{D'} \sum_{S_I} \Delta m \right\}^{1/2} \\ &\leqq \left\{ \sum_{D'} \sum_{S_I} H(J) - 2G(J) + G(J) \right\}^{1/2} \{m(b) - m(a)\}^{1/2} \\ &\leqq (c/4)\{m(b) - m(a)\}^{1/2}. \end{aligned}$$

It therefore follows that $\left| \int_{[a,b]} [dldm]^{1/2} - \int_{[a,b]} [dgdm]^{1/2} \right| \leqq c/8 + (c/4)\{m(b) - m(a)\}^{1/2} + c/16 + (c/4)\{m(b) - m(a)\}^{1/2} + c/8$. Therefore $\int_{[a,b]} [dldm]^{1/2} = \int_{[a,b]} [dgdm]^{1/2}$.

If U is a subdivision of $[a,b]$, then $l(b) - g(b) = \sum_U \Delta l - \Delta g = (\sum_U [\Delta l \Delta l]^{1/2} - \sum_U [\Delta g \Delta l]^{1/2}) + (\sum_U [\Delta l \Delta g]^{1/2} - \sum_U [\Delta g \Delta g]^{1/2})$, so that

$$\begin{aligned} l(b) - g(b) &= \left(\int_{[a,b]} [dldl]^{1/2} - \int_{[a,b]} [dgdl]^{1/2} \right) \\ &\quad + \left(\int_{[a,b]} [dldg]^{1/2} - \int_{[a,b]} [dgdg]^{1/2} \right) = 0. \end{aligned}$$

Therefore $\int_{[a,b]} H(I)$ exists.

References

1. E. Hellinger, *Die Orthogonalinvarianten Quadratischer Formen von unendlichvielen Variablen*, Dissertation, Göttingen, 1907.

2. A. Kolmogoroff, *Untersuchen über den Integralbegriff*, Math. Ann. **103** (1930), 654–696.

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