ON THE CONVERGENCE OF INFINITE EXPONENTIALS

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Given a sequence of complex numbers $\{a_i\}$, we define a sequence of functions:

$$a_1^z, \quad a_1^{(a_2^z)}, \quad a_1^{(a_2^{(a_3^z)})}, \cdots.$$

This sequence is formally represented by $E(a_1, a_2, a_3, \cdots; z)$, and is called an infinite exponential. If all the $a_i = a$, as is the case in this paper, the infinite exponential is represented by $E(a; z)$. This nomenclature follows Barrow [1]. Another symbolism has been used by Thron [9]. It is necessary to specify the values $\log a_n$ to make the sequence determinate.

Euler [4] was the first to investigate seriously the convergence of the sequence $E(a; z)$. He stated and demonstrated the results for the case where a and z are real. However, his demonstration was not a rigorous proof. Later proofs of these results were given independently by Seidel [8], Gravé [5], and Barrow [1]. We quote the result that $E(a; 1)$ converges when

$$e^{-e} \leqq a \leqq e^{1/e}.$$

Convergence for complex a_i has been considered by Thron [9]. He showed that $E(a_1, a_2, a_3, \cdots; 1)$ will converge if $|\log a_i| \leqq e^{-1}$, $i = 1, 2, 3, \cdots$. The purpose of this paper is to establish the existence of a region of convergence extending outside this one in the special case that $a_i \equiv a$. If $E(a; z)$ converges to the limit t, then clearly

$$a^t = t \quad \text{or} \quad a = t^{1/t}.$$

If now one lets $t = e^{\zeta}$ where ζ is a complex parameter, then [2]

$$a = e^{\zeta e^{-\zeta}}. \tag{1}$$

In the remainder of this paper, the representation (1) will be employed.

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$E(a; z)$ is the sequence of iterates of the function $\phi(z)$. That is, it is identical with the sequence $\phi_n(z)$ defined by

$$\phi_1(z) = \phi(z) = a^z,$$
$$\phi_2(z) = \phi(\phi_1(z)) = a^{\phi_1(z)},$$
$$\phi_n(z) = \phi(\phi_{n-1}(z)) = a^{\phi_{n-1}(z)}.$$

Now $\phi(z)$ is an entire function with the fixed point $z_0 = e^{\zeta}$ and the theory of iteration [6; 7, pp. 229–239] tells us that the sequence $\phi_n(z)$ will converge for z in some neighborhood of the fixed point provided that $|\phi'(z_0)| < 1$.

Now using (1)

$$\frac{d\phi}{dz} = \frac{d}{dz} e^{\zeta e^{-\zeta} z} = e^{\zeta e^{-\zeta} z} \zeta e^{-\zeta}, \qquad \frac{d\phi(e^{\zeta})}{dz} = \zeta.$$

Hence we have [2]

THEOREM 1. *If $|\zeta| < 1$ and $a = e^{\zeta e^{-\zeta}}$, then $E(a; z)$ converges to e^{ζ} for z in some neighborhood of $z_0 = e^{\zeta}$.*

The results of this paper are based on the estimate of the convergence neighborhood given in

THEOREM 2. *If $|\zeta| < 1$ and $a = e^{\zeta e^{-\zeta}}$ then $E(a; z)$ converges to e^{ζ} provided $|z - e^{\zeta}| < r_0 |e^{\zeta}|$ where r_0 is the positive root of the equation $r^{-1} \log(1 + r) = |\zeta|$.*

The convergence is uniform in

$$|z - e^{\zeta}| \leqq \theta r_0 |e^{\zeta}|; \qquad 0 < \theta < 1. \tag{2}$$

From the point of view of the theory of iteration of functions $E(a; 1)$ is merely a special case of $E(a; z)$. This was recognized by Euler, but as Condorcet [3] originally raised the question of the convergence of $E(a; 1)$ there is particular interest in determining the values of a for which $E(a; 1)$ converges. We shall deduce from Theorem 2

THEOREM 3. *If $a = e^{\zeta e^{-\zeta}}$ and $|\zeta| \leqq \log 2$ then $E(a; 1)$ converges to e^{ζ}.*

A slightly more extensive region than $|\zeta| \leqq \log 2$ will be found as stated below in Theorem 4.

The regions of convergence of $E(a; 1)$ in the upper half of the a-plane as established by Thron, Theorem 3 and Theorem 4 are illustrated in Figure I which is based on computations performed on an electronic computer.

Theorem 1 was given by Carlsson [2].

Before proceeding to prove Theorem 2 it is helpful to consider the function

$$\theta(r) = \frac{e^{rs} - 1}{r} = s + \frac{rs^2}{2!} + \frac{r^2s^3}{3!} + \cdots .$$

We evidently have the following

LEMMA. *If* $0 < s < 1$ *then* $\theta(r)$ *is monotone increasing with* r *for* $r > 0$. *There exists a unique positive solution* $r = r_0$ *of the equation* $r^{-1} \log (1+r) = s$ *and* $\theta(r_0) = 1$. $s < \theta(r) < 1$ *for* $0 < r < r_0$.

Now let $z = e^{\zeta}(1+\beta)$. We find that $a^z = \exp(z\zeta e^{-\zeta}) = e^{\zeta}e^{\beta\zeta}$. We propose to establish conditions under which $|a^z - e^{\zeta}| < |z - e^{\zeta}|$. Evidently $|a^z - e^{\zeta}| = |e^{\zeta}|\,|e^{\beta\zeta} - 1| \leqq |e^{\zeta}|\,(e^{|\beta\zeta|} - 1)$. Now with $s = |\zeta|$ we find that provided $|\beta| \leqq r$ then

$$|a^z - e^{\zeta}| \leqq |e^{\zeta}|\,\theta(r)\,|\beta| = \theta(r)\,|z - e^{\zeta}|.$$

Hence if $|\beta| \leqq r < r_0$ we have

$$|\phi_1(z) - e^{\zeta}| = |\beta_1 e^{\zeta}| \leqq \theta(r)\,|z - e^{\zeta}| = \theta(r)\,|\beta e^{\zeta}|.$$

This implies that $|\beta_1| \leqq \theta(r)|\beta| \leqq \theta(r)r \leqq r$ and hence repeating the argument

$$|\phi_2(z) - e^{\zeta}| = |\beta_2 e^{\zeta}| \leqq \theta(r)\,|\phi_1(z) - e^{\zeta}| \leqq [\theta(r)]^2\,|z - e^{\zeta}|.$$

Evidently $|\phi_n(z) - e^{\zeta}| \leqq [\theta(r)]^n |z - e^{\zeta}|$ and $\phi_n(z)$ converges uniformly to e^{ζ} for $|z - e^{\zeta}| \leqq r|e^{\zeta}|$. This proves Theorem 2.

We prove Theorem 3 by showing that $z = 1$ is within the circle of convergence associated with each number a in the relevant region. Substituting $|\zeta| = \log 2$ in equation (2) we find $r_0 = 1$. Hence, we shall have convergence in this case if $|\beta| < 1$. The point under consideration is $z = 1 = e^{\zeta}(1+\beta)$. Here $\beta = e^{-\zeta} - 1$ and $|\beta| = |e^{-\zeta} - 1| \leqq e^{|\zeta|} - 1 \leqq e^{\log 2} - 1 = 1$. Now the equality holds only when $-\zeta = |\zeta| = \log 2$. Otherwise, $|\beta| < 1$. But when $\zeta = -\log 2$, we appeal to the results for real a, which indicate convergence when $a = 1/4$ (corresponding to $\zeta = -\log 2$). This proves Theorem 3.

It is clear that equation (2) assures convergence in a somewhat larger region. Thus letting

$$r = |\beta| = |e^{-\zeta} - 1|$$

we define a contour in the ζ-plane, namely

$$|\zeta| = \frac{\log(1 + |e^{-\zeta} - 1|)}{|e^{-\zeta} - 1|} \tag{3}$$

within which $|e^{\zeta}-1|<r_0|e^{\zeta}|$. Hence, we have

THEOREM 4. *The infinite exponential* $E(a;\,1)$ *converges to* e^{ζ} *when* ζ *is within the contour defined by equation* (3) *and* $a=e^{\zeta e^{-\zeta}}$.

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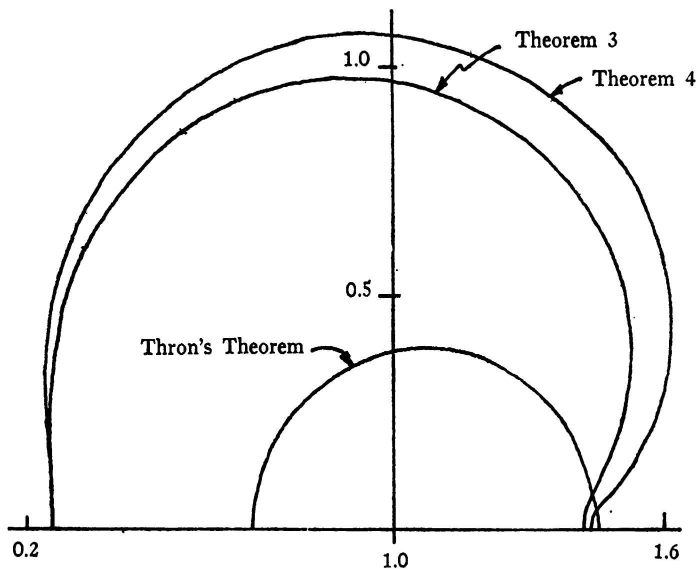

FIGURE 1. Regions of convergence of $E(a;\,1)$.

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