THE EXISTENCE OF COMPACT LINEAR MAPS BETWEEN BANACH SPACES
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In [2] J. D. Weston proves that given any separable Banach space Y, there exist a normed linear space X and a compact one-one linear operator which maps the conjugate space X' onto a subspace dense in Y.

It is the purpose of this paper to solve the following problem.

Given normed linear space X and Banach space Y, under what conditions does there exist a one-one compact linear map from X onto a subspace dense in Y?

Necessary and sufficient conditions for such an operator to exist are given (cf. (C) of the theorem below).

We first introduce some notations.

Suppose X and Y are normed linear spaces. Then $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$) is the space of all bounded (resp., compact) linear maps from X to Y. $\mathcal{B}_1(X, Y)$ (resp., $\mathcal{K}_1(X, Y)$) is the set of all one-one maps in $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$). $\mathcal{B}_d(X, Y)$ (resp., $\mathcal{K}_d(X, Y)$) is the set of all maps in $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$) with range dense in Y. $\mathcal{B}_{1,d}(X, Y) = \mathcal{B}_1(X, Y) \cap \mathcal{B}_d(X, Y)$, and $\mathcal{K}_{1,d}(X, Y) = \mathcal{K}_1(X, Y) \cap \mathcal{K}_d(X, Y)$.

Finally, $\emptyset$ is the void set.

If X is a normed linear space and A is a subset of X', then A is total if and only if for each $x \neq 0$ in X there exists an $\alpha \in A$ such that $\alpha x \neq 0$. The following preliminary remarks are easily verified. The first of these gives alternative ways of stating a condition arising prominently in the rest of the paper.

(i) If X is a normed linear space, then X' contains a countable total subset if and only if X' is separable with respect to the $w^*$ topology and also if and only if X' contains a total separable linear subspace.

(ii) If X is a separable normed linear space, then each of the conjugate spaces X' and X'' contains a countable total subset.

(iii) If X and Y are normed linear spaces, $\mathcal{B}_d(X, Y) \neq \emptyset$, and Y' has a countable total subset, then X' has a countable total subset.

Definition. Suppose X is a Banach space, and suppose $x_k$ is in X and $\varepsilon_k$ is a real number for $k = 1, 2, \ldots$. Then \{$x_k\}_{i=1}^\infty$ is $\varepsilon$-independent if and only if

\[ \sum_{k=1}^\infty \| x_k \| < \infty \]

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(b) for each bounded sequence \( \{\alpha_k\}_1^\infty \) of scalars, \( \sum_{k=1}^\infty \alpha_k x_k = 0 \) implies \( \alpha_k = 0 \) \( (k=1, 2, \cdots) \).

Remark. \( \epsilon_k \neq 0 \) \( (k=1, 2, \cdots) \) if \( \{x_k\}_1^\infty \) is \( \{\epsilon_k\}_1^\infty \)-independent.

Lemma. Suppose \( \{x_k\}_1^\infty \) is a linearly independent sequence of elements in a Banach space \( X \). Then there exists a sequence \( \{\epsilon_k\}_1^\infty \) of positive real numbers such that \( \{x_k\}_1^\infty \) is \( \{\epsilon_k\}_1^\infty \)-independent.

Proof. By considering the sequence \( \{x_k/\|x_k\|\}_1^\infty \), we may assume \( \|x_k\| = 1 \) \( (k=1, 2, \cdots) \).

For each positive integer \( n \), let \( l(n) \) be the Banach space of \( n \)-tuples of scalars with norm defined by \( \| (\eta_1, \eta_2, \cdots, \eta_n) \| = \sum_{k=1}^n |\eta_k| \). The map \( f \) defined by \( f(\eta_1, \eta_2, \cdots, \eta_n) = \| \sum_{k=1}^n \eta_k x_k \| \) is a continuous map from \( l(n) \) into the space of real numbers. Define \( S_n \) to be the set of all \( (\eta_1, \eta_2, \cdots, \eta_n) \) in \( l(n) \) such that \( 1/2 \leq \| (\eta_1, \eta_2, \cdots, \eta_n) \| \leq n \). Then \( S_n \) is a compact subset of \( l(n) \). Hence \( f \) attains a minimum \( a_n \) on \( S_n \). Since \( \{x_k\}_1^\infty \) is linearly independent, \( a_n > 0 \). Clearly, \( 1 > a_n \geq a_{n+1} \) \( (n=1, 2, \cdots) \).

Let \( \epsilon_1 = 1 \). Define \( \epsilon_{n+1} = 2^{-n} a_{n+1} \) for \( n=1, 2, \cdots \). Now

\[
0 < \epsilon_{n+j} \leq (2^{-j})^n \epsilon_n a_n \leq 2^{-nj} \quad (j, n = 1, 2, \cdots).
\]

Consider a bounded sequence \( \{\alpha_k\}_1^\infty \) of scalars not all 0. Let \( \alpha = \sup_{k=1}^\infty |\alpha_k| \). Then \( \alpha > 0 \), and there exists an integer \( N \) such that \( |\alpha_N| > \alpha/2 \). Suppose \( |\epsilon_{n} \alpha_{n} \| = \max_{j=1}^N |\epsilon_{j} \alpha_{j} \| \). Then \( |\epsilon_{n} \alpha_{n} \| \geq |\epsilon_{N} \alpha_{N} \| > \epsilon_{N} \alpha/2 > 0 \), and

\[
\left| \sum_{j=1}^N \epsilon_{j} \alpha_{j} x_{j} \right| = \left| \epsilon_{n} \alpha_{n} \| \sum_{j=1}^N \epsilon_{j} \alpha_{j} x_{j}/(\epsilon_{n} \alpha_{n} \|) \right|
\geq \left| \epsilon_{n} \alpha_{n} \| a_n > \alpha \epsilon_N a_N/2. \right.
\]

From (1) we have

\[
\left| \sum_{j=1}^\infty \epsilon_{j} \alpha_{N+j} x_{N+j} \right| \leq \alpha \sum_{j=1}^\infty \epsilon_{j} \alpha_{j} x_{j} \leq a \sum_{j=1}^\infty 2^{-nj} \epsilon_{N} a_N < \alpha \epsilon_N a_N/4.
\]

By (2) and (3),

\[
\left| \sum_{j=1}^\infty \epsilon_{j} \alpha_{j} x_{j} \right| \geq \left| \sum_{j=1}^N \epsilon_{j} \alpha_{j} x_{j} \right| - \left| \sum_{j=1}^\infty \epsilon_{j} \alpha_{N+j} x_{N+j} \right| > (\alpha \epsilon_N a_N/2) - (\alpha \epsilon_N a_N/4) = \alpha \epsilon_N a_N/4 > 0.
\]

Thus (b) is proved. That (a) holds follows from the fact that

\[
0 < \sum_{k=1}^\infty \epsilon_{k} \leq \sum_{j=0}^\infty 2^{-nj} < \infty.
\]
Theorem. Suppose $X$ is an infinite-dimensional normed linear space and $Y$ is an infinite-dimensional Banach space. Then (A)-(C) below hold.

(A) $\mathcal{K}_s(X, Y) \neq \emptyset$ if and only if $X'$ has a denumerable total subset.

(B) $\mathcal{K}_d(X, Y) \neq \emptyset$ if and only if $Y$ is separable.

(C) $\mathcal{K}_{s,d}(X, Y) \neq \emptyset$ if and only if $\mathcal{K}_s(X, Y) \neq \emptyset$ (alternatively, $\mathcal{K}_d(X, Y) \neq \emptyset$) and $\mathcal{K}_d(X, Y) \neq \emptyset$, i.e., if and only if $Y$ is separable and $X'$ has a denumerable total subset.

Moreover, each of the sets $\mathcal{K}_s(X, Y)$, $\mathcal{K}_d(X, Y)$, $\mathcal{K}_{s,d}(X, Y)$ which is nonvoid contains a map which is the limit (in norm) of continuous linear maps having finite-dimensional range.

Proof. There exist linearly independent sequences $\{x'_k\}^\infty_k$ and $\{y_k\}^\infty_k$ in $X'$ and $Y$ respectively such that $\|x'_k\| = \|y_k\| = 1$ $(k = 1, 2, \ldots)$. In addition, we may take $\{x'_k\}^\infty_k$ total in $X'$ if $X'$ has a denumerable total subset, and we may take $\{y_k\}^\infty_k$ spanning a dense subset of $Y$ if $Y$ is separable. By the lemma, $\{x'_k\}^\infty_k$ is $\{e'_k\}^\infty_k$-independent and $\{y_k\}^\infty_k$ is $\{e_k\}^\infty_k$-independent for some $\{e'_k\}^\infty_k$ and $\{e_k\}^\infty_k$. Let $T : X \to Y$ and $T_n : X \to Y$ be defined by $Tx = \sum_{k=1}^n e_kx'_k(x)y_k$ and $T_n x = \sum_{k=1}^n e_kx'_k(x)y_k$. Clearly, $T_n$ is a bounded linear operator with finite-dimensional range and hence is compact. Moreover, $\{T_n\}^\infty_n$ converges in norm to $T$, for if $x$ is in $X$ and $\|x\| = 1$, then $\|T_n x - Tx\| \leq \sum_{n=1}^\infty \|e_k\| \|x'_k\| \|y_k\| \|e_k\| \|x'_k\| \|y_k\|$. Therefore $T$ is a compact linear operator.

Suppose $X'$ has a denumerable total subset. Consider $x$ in $X$ such that $Tx = 0$. Since $\{y_k\}^\infty_k$ is $\{e_k\}^\infty_k$-independent, $e_kx'_k(x) = 0$ and hence $x'_k(x) = 0$ $(k = 1, 2, \ldots)$. Therefore $x = 0$ since $\{x'_k\}^\infty_k$ is total in $X'$. Thus $T$ is one-one, $T$ is in $\mathcal{K}_s(X, Y)$, and $\mathcal{K}_s(X, Y) \neq \emptyset$.

Suppose $Y$ is separable. To show that $(TX)^\perp = Y$, suppose the contrary. Then there exists $y' \neq 0$ in $Y'$ such that $0 = y'TX$, whence for each $x$ in $X$,

$$0 = y'\left(\sum_{k=1}^\infty e_k\eta_kx'_k(x)y_k\right) = \sum_{k=1}^\infty e_k\eta_ky'(y_k)x'_k(x).$$

Therefore $\sum_{k=1}^\infty e_k\eta_ky'(y_k)x'_k = 0$. Since $\{x'_k\}^\infty_k$ is $\{e'_k\}^\infty_k$-independent, it follows that $y'y_k = 0$ $(k = 1, 2, \ldots)$. Hence $y' = 0$ since $\{y_k\}^\infty_k$ generates a subspace dense in $Y$. Thus we have contradicted the statement that $y' \neq 0$. Since $(TX)^\perp = Y$, $T$ is in $\mathcal{K}_d(X, Y)$, and $\mathcal{K}_d(X, Y) \neq \emptyset$.

If $X'$ has a denumerable total subset and $Y$ is separable, then $T$ is in $\mathcal{K}_s(X, Y) \cap \mathcal{K}_d(X, Y) = \mathcal{K}_{s,d}(X, Y)$, and $\mathcal{K}_{s,d}(X, Y) \neq \emptyset$.

Half of each of (A)-(C) has been proved. The other half of (A) follows from (iii) with the use of (ii) and the observation that for each $T$ in $\mathcal{K}_s(X, Y)$, $T(A)$ is separable and $T$ is in $\mathcal{K}_d(X, T(A))$. The other
half of (B) follows from the fact that the range of each map in \( \mathcal{K}(X, Y) \) is separable. The other half of (C) follows from (A) and (B). The last statement of the theorem is now clear from the proof thus far.

**Corollary.** Suppose \( X \) and \( Y \) are separable infinite-dimensional normed linear spaces with \( Y \) a Banach space. Then \( \mathcal{K}_{\sigma,e}(X, Y) \) and \( \mathcal{K}_{\sigma,e}(X', Y) \) are nonempty.

**Proof.** Apply (ii) and the theorem.

The proof of the theorem leads to the following observations.

**Observations.** Suppose \( X \) is an infinite-dimensional normed linear space and \( Y \) is an infinite-dimensional separable Banach space. Then (A')--(C') below hold.

(A') \( \mathcal{B}_0(X, Y) \neq \emptyset \) if and only if \( X' \) has a denumerable total subset.

(B') \( \mathcal{B}_0(X, Y) \neq \emptyset \).

(C') \( \mathcal{K}_{\sigma,e}(X, Y) \neq \emptyset \) if and only if \( \mathcal{B}_0(X, Y) \neq \emptyset \), i.e., if and only if \( X' \) has a denumerable total subset.

One could remove “infinite-dimensional” from the hypotheses of the theorem and the observations and still obtain “if and only if” results similar to, but slightly more complicated than, (A)--(C) and (A')--(C'). The details are completely straightforward.

Statement (A') will now be used to generalize a theorem of J. A. Clarkson.

A norm \( \| \| \) on a linear space \( X \) is called **strictly convex** if and only if \( \| x + y \| < \| x \| + \| y \| \) for all \( x \) and \( y \) in \( X \) which are linearly independent. Suppose \( X \) and \( Y \) are normed linear spaces with the norm on \( Y \) strictly convex, and suppose \( T \) is in \( \mathcal{B}_0(X, Y) \). The norm \( \| \| ' \) on \( X \) such that \( \| x \| ' = \| x \| + \| Tx \| \) for each \( x \) in \( X \) is strictly convex and is equivalent to the original norm on \( X \). Hence, applying (A') with \( Y \) a Hilbert space, one obtains immediately the following result, whose special case resulting from taking \( X \) separable is due to J. A. Clarkson [1, Theorem 9].

**Proposition.** Suppose \( X \) is a normed linear space whose conjugate space has a countable total subset. (For example, suppose \( X \) is a separable normed linear space or the conjugate space of a separable normed linear space.) Then the norm of \( X \) is equivalent to a strictly convex norm.

**References**


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