ISOMETRIC IMMERSIONS WHICH PRESERVE CURVATURE OPERATORS

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The curvature tensor of a Riemannian manifold $M$ can be expressed by a function which assigns to each pair of vectors $x, y \in M_m$ (tangent space to $M$ at $m$) a skew-symmetric linear operator $R_{xy}$ on $M_m \ [1]$. Call $R_{xy}$ the curvature operator of $x, y$. Let $j: M^d \rightarrow \overline{M}^{d+1}$ be an isometric immersion. If $j$ is totally geodesic, then $j$ preserves curvature operators, that is, if $x, y, z \in M_m$, then $d j(R_{xy}(z)) = R_{dx}(j(y), j(z))$. The converse is generally false. We are going to consider the character of immersions as above which preserve curvature operators. The simplest example is an arbitrary isometric immersion of $R^d$ in $R^{d+1}$. In particular we show that if the domain $M^d$ of $j$ is complete and has positive curvature then the converse above holds, that is, if $j$ preserves curvature operators, then $j$ is totally geodesic.

1. General case. Note that $j: M^d \rightarrow \overline{M}^{d+1}$ preserves curvature operators if and only if (a) $j$ preserves Riemannian curvature, i.e., $\overline{K}(dj(\pi)) = K(\pi)$ for all 2-planes $\pi$ tangent to $M$, and (b) if $z \in M_{j(m)}$ is orthogonal to $d j(M_m)$, then $\overline{R}_{d j(\pi)}(z) = 0$ for all $x, y \in M_m$. The proof is elementary, and depends on the fact that the codimension of $M$ in $\overline{M}$ is one.

Theorem 1. Let $j: M^d \rightarrow \overline{M}^{d+1}$ be an isometric immersion which preserves curvature operators, and let $M$ be complete. Then the open set $N$ of nongeodesic points of $M$ rel. $j$ is foliated by complete $(d-1)$-dimensional submanifolds which are totally geodesic rel. $j$.

Proof. Since $j$ preserves Riemannian curvature, at each point of $M$ there is at most one curvature direction with nonzero principal curvature. Thus on the set $N$ of nongeodesic points, the directions of zero normal curvature constitute a differentiable field $\Phi$ of $(d-1)$-planes. We will integrate $\Phi$ to obtain the required foliation. (The theorem holds trivially when $N$ is empty.)

Each point of $N$ has a neighborhood $U$ on which there is a unit normal vector field $E_{d+1}$ rel. $j$ and a frame field $E = (E_1, \ldots, E_d)$ whose first vector is in the curvature direction with principal curvature $\kappa_1 \neq 0$. From the frame field $E$ one obtains on $U$ the dual-base forms $\omega_i$, the Riemannian connection forms $\phi_{ij}$, and curvature forms $\Phi_{ij}$ of $M$, $1 \leq i, j \leq d$. Enlarging $E$ by adding $E_{d+1}$ to it, we get the Codazzi forms $\sigma_i$, $1 \leq i \leq d$, and curvature forms $\Phi_{rt}$, $1 \leq r, s \leq d+1$.
of $M$. Dropping the differential map of $j$ from the notation, we can write
\[ R_{E_i E_j}(E_{d+1}) = - \sum_k \Phi_{k,d+1}(E_i, E_j) E_k. \]
Thus by (b) above, we have $\Phi_{k,d+1} = 0$ on $U$. Furthermore, $\sigma_1 = \kappa \omega_1 \neq 0$, and $\sigma_i = 0$ if $i > 1$.
Thus the Codazzi equations $d\sigma_i = - \sum \phi_{ik} \wedge \sigma_k - \Phi_{d+1, i}$ reduce to $d\sigma_1 = 0$ and $\phi_{11} \wedge \sigma_1 = 0$. Since $\sigma_1$ annihilates the planes of $\sigma$, $d\sigma_1 = 0$ implies $\sigma$ is integrable. The other equations imply that the forms $\phi_{ii}$ are zero on vectors tangent to a leaf $L$ of $\sigma$. But these forms, $1 < i \leq d$, are the Codazzi forms for $L$ in $M$, so each leaf $L$ is totally geodesic in $M$—and hence also in $M$, i.e. rel. $j$.

Now we show that the leaves $L$ are complete by showing that geodesics of $L$ are infinitely extendible. Suppose the contrary, i.e. that there is a maximal geodesic $\alpha$ of a leaf $L$ which is defined only on a bounded open interval $(a, b)$. Since $M$ is complete, $\alpha$ is infinitely extendible as a geodesic of $M$. Since $L$ is totally geodesic, as long as this extension $\tilde{\alpha}$ remains in $N$, it is a geodesic of $L$. So the limit points $\tilde{\alpha}(a)$ and $\tilde{\alpha}(b)$ of $\alpha$ are not in $N$. We will contradict this by showing that $\kappa_1$ is zero at neither of these points. We can assume that the geodesic segment $\alpha$ (but not its limit points) lies in the domain of fields $E$ and $E_{d+1}$ as above, with the further properties that $\alpha$ is an integral curve of $E_2$ and that $E$ is parallel on $\alpha$. In fact, once $E$ is properly defined on $\alpha$, one can extend over a neighborhood of $\alpha$ in $M$ by first extending over a neighborhood in the leaf $L$, keeping $E_1$ perpendicular to $L$, then extending over the full neighborhood, keeping $E_1$ always in the $\kappa_1$ curvature direction. (Strictly speaking, one passes to a suitable covering manifold if $\alpha$ crosses itself.)

From the first structural equation, we deduce $[E_1, E_2] = \sum \phi_{12}(E_i) E_i$. Applying the form $d\sigma_1 = 0$ to the fields $E_1, E_2$ gives $E_2(\kappa_1) = -\kappa_1 \phi_{12}(E_1)$. Setting $k = \kappa_1 \circ \alpha$, $f = \phi_{12}(E_1) \circ \alpha$, we write this equation as

$$k' = -kf.$$  

Applying the second structural equation to the fields $E_1, E_2$ and simplifying, using the facts above, we get $E_2(\phi_{12}(E_1)) = -f^2 - \Phi_{12}(E_1, E_2)$. Setting $F = \Phi_{12}(E_1, E_2) \circ \alpha$ yields

$$f' = -f^3 - F.$$  

Our assumption that $L$ is not complete has led to the conclusion that $k(t)$ approaches zero as $t$ approaches either $a$ or $b$. The differential equations (1) and (2) contradict this. In fact, solving (1) explicitly, we deduce that as $t \to b$, $\limsup f = +\infty$. This contradicts (2) which says, since $F$ is bounded below on $(a, b)$, that when $f$ is large enough its slope is negative. The argument when $t \to a$ is similar, so the proof is complete.
A scheme similar to that above was used by Chern and Lashof in [3, Lemma 2].

**Theorem 2.** Suppose $M^d$ ($d \geq 2$) is complete and has Riemannian curvature $K > 0$. Then every isometric immersion $j: M^d \to \overline{M}^{d+1}$ which preserves curvature operators is totally geodesic.

**Proof.** Suppose there is a nongeodesic point, that is (in the notation of the previous proof) $N$ is not empty. Then a geodesic $\alpha$ as in that proof has domain the whole real line. Thus we can arrange for the function $f = \phi_{2a}(E_1) \circ \alpha$ to be defined on the whole real line, and $f$ satisfies the differential equation (2) $f'' = -f^2 - F$. But this is impossible when $K > 0$, since then $F > 0$.

This is not a local result—it fails if $M$ is not required to be complete.

2. Constant curvature case. If $\overline{M}^{d+1}$ has constant curvature, then its curvature operators have the property that $R_{zv}(z) = 0$ if $z$ is perpendicular to $x$ and $y$. (Converse, §177 of [2].) Thus by the first remark of the previous section, if $M^d$ and $\overline{M}^{d+1}$ have the same constant curvature, then every isometric immersion $j: M^d \to \overline{M}^{d+1}$ preserves curvature operators. We consider the character of $j$ and $M^d$ when $\overline{M}^{d+1}$ is specialized to be a sphere $S^{d+1}(C)$, Euclidean space $R^{d+1}$, or hyperbolic space $Q^{d+1}(C)$, where $C$ is curvature of appropriate sign. From Theorem 2 we get: if $M^d$ is complete and has constant curvature $C > 0$, then $M^d$ can be immersed in $S^{d+1}(C)$ if and only if $M^d$ is isometric to $S^d(C)$. Any such immersion is an imbedding onto a great $d$-sphere.

In the case $C = 0$, Hartman and Nirenberg [4] have proved: a complete flat manifold $M^d$ can be immersed in $R^{d+1}$ if and only if $M^d$ is isometric to either $R^d$ or $S^1(r) \times R^{d-1}$. Any such immersion is as a cylinder in $R^{d+1}$.

This can be proved by applying Theorem 1 to both $j: M^d \to R^{d+1}$ and $j \circ \pi: R^d \to R^{d+1}$, where $\pi: R^d \to M^d$ is the universal covering of $M^d$. The special character of disjoint, totally geodesic hypersurfaces in $R^d$ allows us to extend the foliation of the set $N$ in $R^d$ to a foliation of all of $R^d$ by parallel $(d-1)$-planes.

This general scheme fails in the negative curvature case, since disjoint, totally geodesic hypersurfaces in $Q^d(C)$ can have more complicated arrangements. One can exhibit surfaces with curvature $C < 0$ in $Q^d(C)$ with arbitrary first Betti number. However the Euclidean result can be extended topologically to the negative curvature case as follows:
Theorem 3. Let \( M^d \) be a complete manifold with constant negative curvature \( C \). If \( M^d \) can be isometrically immersed in \( Q^{d+1}(C) \), then \( H^i(M^d) = 0 \) for \( i \geq 2 \).

(Here \( H \) denotes Čech cohomology with arbitrary coefficients.)

Proof. From such an immersion \( j \) we get a decomposition of \( M \) as in Theorem 1. Denote the components of \( N \) by \( N_a \), the components of \( M - N \) by \( F_\beta \). Each leaf \( L \) of \( N \) is complete and totally geodesic rel. \( j \), hence isometric to \( Q^{d-1} = Q^{d-1}(C) \). The immersion \( j \) is one-to-one on components \( F_\beta \) also. Let \( \pi : Q^d \to M^d \) be the universal covering. Then we can derive

1. If a subset \( A \) of \( M \) can be lifted into \( Q^d \), so can the union of those sets \( L \) and \( F_\beta \) which meet \( A \).

2. There is a number \( \epsilon > 0 \) such that if \( B, C, D \) are disjoint totally geodesic hypersurfaces in \( Q^d \) which meet an \( \epsilon \)-neighborhood, then \( B, C, D \) are linearly ordered, i.e. some one separates the other two in \( Q^d \).

3. Each \( F_\beta \) is either a totally geodesic \( Q^{d-1} \) or (if its interior is not empty) a manifold with boundary \( B_\beta \), where \( B_\beta \) is a union of totally geodesic sets \( Q^{d-1} \), each of which is disjoint from the closure of the others. In particular each \( F_\beta \) is contractible.

By a theorem of Ricci (§107, [2]) the orthogonal trajectories of the leaves of an \( N_a \) give isometries of the leaves. If \( N \) is dense in \( M \) it follows (much as in the Euclidean case) that \( M \) is diffeomorphic to either \( R^d \) or \( S^1 \times R^{d-1} \). Excluding this case we have

4. The boundary of each \( N_a \) is either a single totally geodesic \( Q^{d-1} \) or two disjoint ones, and the closure \( \overline{N}_a \) of \( N_a \) is contractible.

Consider the covering \( \mathcal{C} \) of \( M \) by all sets \( N_a \) and \( F_\beta \). This is a closed covering by homologically trivial sets. Furthermore, any intersection of three elements of \( \mathcal{C} \) is empty, and the intersection of any two consists of at most two disjoint sets \( Q^{d-1} \). Suppose \( \mathcal{C} \) is locally finite, e.g. \( M - N \) only a finite number of components. Then by a well-known theorem, the cohomology of \( M \) is isomorphic to the cohomology of the nerve of \( \mathcal{C} \). Since this nerve has dimension 1 the result follows. If \( \mathcal{C} \) is not locally finite we can alter it, retaining its essential properties, so as to get local finiteness. We omit the details of the proof. Roughly speaking, if \( \mathcal{C} \) is not locally finite at a point \( p \), then \( p \) lies in a "limit face" \( Q_1 \) of an element, say \( \overline{N}_a \), of \( \mathcal{C} \). Choose \( N_\beta \neq N_a \) sufficiently near \( Q_1 \) and let \( Q_2 \) be the face of \( N_\beta \) nearest \( Q_1 \). Using (1) and (2) we can define \( G \) to be the union of \( Q_1, Q_2 \), and the elements of \( \mathcal{C} \) between \( Q_1 \) and \( Q_2 \). Finally, replace these elements by \( G \) in \( \mathcal{C} \). Iteration of this operation eliminates all limit faces.

In general the complexity of the decomposition of \( M \) given by
Theorem 1 is measured by the identification space $M^*$ whose elements are the leaves of $N$ and the components of $M - N$. If $M^*$ is metrizable, it can be shown to have inductive dimension 1. In this case the argument above can be replaced by an application of the Vietoris mapping theorem.

Bibliography


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ON THE EMBEDDABILITY OF THE REAL PROJECTIVE SPACES

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In a paper of the same title, Massey [4] proved that if $2^{k-1} + 2^{k-1} - 1 \leq n < 2^k$ then $P_n$ cannot be differentiably embedded in $\mathbb{R}^k$. By using the technique of Massey in a different way we can prove the following theorem which clearly includes Massey's.

**Theorem.** If $2^{k-1} < n < 2^k$ then $P_n$ cannot be embedded differentiably in Euclidean space of dimension $2^k$.

Besides the result of Massey, the main result in this direction is if $2^{k-1} < n < 2^k$ then $P_n$ cannot be embedded differentiably in $\mathbb{R}^{2k-1}$. Our result yields, in particular, that for $P^{2k+1}$, the embedding in $\mathbb{R}^{2k+1}$ given by Hopf and James [1] is the best possible.

The following information from [3; 4] will be needed. Let $M$ be an $n$-manifold differentiably embedded in $\mathbb{R}^{n+k+1}$; and let $p: E \to M$ denote the bundle of unit normal vectors. Then there exist subalgebras $A^*(E, Z) \subset H^*(E, Z)$ and $A^*(E, Z_2) \subset H^*(E, Z_2)$ which satisfy the following conditions:

1. $A^q(E, G) = H^q(E, G)$,
2. $H^q(E, G) = A^q(E, G) + p^*(H^q(B, G))$ (0 < $q$ < $n+k$),
3. $A^q(E, G) = 0$, $q \geq n+k$.

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2 The referee has informed me that the result of this paper has been obtained independently by Mr. J. P. Levine in his thesis at Princeton University.