REGULARLY ORDERED GROUPS

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1. **Introduction.** In [**2**] Robinson and Zakon present a metamathematical study of regularly ordered abelian groups, and in [**3**] Zakon proves the theorems that are stated without proofs in [**2**] and further develops the theory of these groups by algebraic methods. It is apparent from either paper that there is a close connection between regularly ordered groups and divisible ordered groups. In this note we establish this connection. For example, an ordered group G, with all convex subgroups normal, is regularly ordered if and only if G/C is divisible for each nonzero convex subgroup C of G. If H is an ordered group with a convex subgroup Q that covers 0, then H is regular if and only if H/Q is divisible. Thus a regularly and discretely ordered abelian group is an extension of an infinite cyclic group by a rational vector space. In any case a regularly ordered group is "almost" divisible. In addition we use the notion of $\mathfrak{J}$-regularity, which is slightly weaker than regularity, and our results are not restricted to abelian groups.

Throughout this paper all groups, though not necessarily abelian, will be written additively, R will denote the group of all real numbers with their natural order, and G will denote a linearly ordered group (notation o-group). A subset S of G is *convex* if the relations $a<x<b$, $a, b\in S$, $x\in G$ always imply that $x\in S$. A subgroup of G with this property is called a *convex subgroup*. G is said to be *regular* if for every infinite convex subset S of G and every positive integer n, there exists an element g in G such that $ng\in S$. Clearly a divisible o-group is regular. If $a, b\in G$ and $a<b$, then $\{x\in G: a<x<b\}$ is the *interval* determined by a and b, and we shall denote this interval by $(a<b)$. G is said to be $\mathfrak{J}$-*regular* if for every infinite interval $(a<b)$ in G and every positive integer n, there exists an element g in G such that $ng\in(a<b)$. It is clear that if G is regular, then it is $\mathfrak{J}$-regular. Also, if G is densely ordered and $\mathfrak{J}$-regular, then it is regular. For in this case every infinite convex subset of G contains an infinite interval. In §4 we give examples of abelian o-groups that are $\mathfrak{J}$-regular, but not regular.

Zakon shows (Theorem 2.4 in [**3**]) that *an Archimedean o-group A is regular and hence $\mathfrak{J}$-regular*. This is clear if A is cyclic, and if A is not cyclic, then (without loss of generality) nA is dense in R for all positive integers n, and so A is regular.

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2. **The relationship between regularity, $\mathfrak{J}$-regularity and divisibility.** Let Γ be the set of all ordered pairs (G^γ, G_γ) of convex subgroups of G such that G^γ covers G_γ. Each G^γ/G_γ is an Archimedean o-group, and hence is o-isomorphic to a subgroup of R, and is regularly ordered. Define that $(G^\alpha, G_\alpha) > (G^\beta, G_\beta)$ if $G_\alpha \supseteq G^\beta$. This is a linear ordering of Γ, and the *rank* of G is the order type of Γ. In particular, G has rank 1 if and only if G is Archimedean. In Propositions 1 through 4 and their corollaries *let C be a nonzero normal convex subgroup of G.* It is easy to show that if G is regular ($\mathfrak{J}$-regular), then C is regular ($\mathfrak{J}$-regular) and if G is $\mathfrak{J}$-regular, then G/C is regular, but we do not need these results.

Proposition 1. *If G is regular, then G/C is divisible.*[1]

Proof. It suffices to show that $C+nG\supseteq G$ for all positive integers n. But for each g in G, the set $C-g$ is infinite and convex and hence meets nG. Therefore $g\in C+nG$.

Proposition 2. *If G is $\mathfrak{J}$-regular, then G/C is divisible except possibly when C is discretely ordered of rank* 1, *in which case G/C is densely ordered.*

Proof. If C is densely ordered or the rank of C exceeds 1, then C contains an infinite interval $(0<c)$. For each $g\in G$ and $n>0$ the interval $(g<g+c)$ is infinite and hence $g<nx<g+c$ for some x in G. Therefore $C+g=C+nx=n(C+x)$, and it follows that G/C is divisible.

Next assume that C is discretely ordered and has rank 1. If G/C has no convex subgroup that covers 0, then clearly G/C is densely ordered. If Q is the convex subgroup of G/C that covers 0, then $Q=C'/C$, where C' is the convex subgroup of G that covers C, and it suffices to show that C'/C is densely ordered. If C'/C is discretely ordered, then C' is o-isomorphic to a direct sum $I\oplus I$ of integers that is lexicographically ordered, say from the right [1, p. 39]. The interval $((0, 1)<(0, 2))$ is infinite and consists of the elements

$$\{(n, 1) \quad \text{and} \quad (-n, 2)\colon n > 0\}.$$

Clearly this interval contains no elements of the form $3(x, y)$, and since C' is convex in G it follows (without loss of generality) that $((0, 1)<(0, 2))$ is an infinite interval in G and that $((0, 1)<(0, 2))$ $\cap 3G$ is the null set, but this contradicts the assumption that G is $\mathfrak{J}$-regular. Therefore C'/C is densely ordered and so is G/C.

[1] The author wishes to thank the referee for the proof of Proposition 1.

PROPOSITION 3. *If every convex subgroup of G is normal, and if G satisfies both of the following conditions, then G is $\mathfrak{I}$-regular.*

(a) *If C has rank 1, then G/C is densely ordered.*

(b) *If the rank of C exceeds 1 or if C is densely ordered, then G/C is divisible.*

PROOF. Let $(a<b)$ be an infinite interval in G, where $a \geqq 0$, and let n be a positive integer. Let C' be the intersection of all convex subgroups of G that contain $a-b$, and let C be the join of all convex subgroups of G that do not contain $a-b$. It follows that C' covers C, $a-b \in C' \backslash C$ and $C+a<C+b$. If $C \neq 0$, then by (a) and (b) C'/C is densely ordered and by (b) G/C' is divisible. If $C=0$, then $(0<b-a)$ is an infinite interval in the Archimedean group C', and hence C' is densely ordered. Thus, by (b), G/C' is divisible. Therefore in any case C'/C is densely ordered and $nd \equiv a \equiv b \bmod C'$ for some d in G. Let $D=C+d$, $Y \in C'/C$, and assume (without loss of generality) that C'/C is a subgroup of R. Then $D+Y-D=kY$ for some $0<k \in R$, because all order preserving automorphisms of a subgroup of R have this form, and by induction

$$n(Y+D) = (k^{n-1}+k^{n-2}+\cdots+k+1)Y+nD.$$

Note that $r=k^{n-1}+\cdots+1$ is a positive real number that depends only on n and D, and that

$$C+a<n(Y+D)<C+b \tag{1}$$

if and only if

$$C+a-nD<rY<C+b-nD. \tag{2}$$

Since C'/C is densely ordered, it is dense in R, but this means that $r(C'/C)$ is dense in R, and hence $r(C'/C)$ is dense in C'/C. It follows that there exists an element Y in C'/C that satisfies (2) and hence (1). $n(Y+D)=n(C+d')=C+nd'$, where $d' \in G$, and hence $a<nd'<b$. Therefore G is $\mathfrak{I}$-regular.

COROLLARY. *Let Q and Q' be convex subgroups of the discretely ordered group G such that Q covers 0 and Q' covers Q. If G/Q is densely ordered and G/Q' is divisible, then G is $\mathfrak{I}$-regular.*

PROOF. Since inner automorphisms preserve order, Q and Q' are normal. Let $(a<b)$ be an infinite interval in G, with $a \geqq 0$, and let n be a positive integer. If $Q'+a<Q'+b$, then since G/Q' is divisible, $Q'+a<Q'+ng<Q'+b$ for some $g \in G$, and hence $ng \in (a<b)$. If $a-b \in Q$, then $(0<b-a)$ is an infinite interval in the cyclic group Q,

which is impossible. Finally, if $a-b\in Q'\backslash Q$, then by the argument in the proof of Proposition 3 (where $C'=Q'$ and $C=Q$), $a<nd'<b$ for some $d'\in G$. Therefore G is $\mathfrak{J}$-regular.

PROPOSITION 4. *If every convex subgroup of G is normal, and if G/C is divisible for all nonzero convex subgroups C of G, then G is regular.*

PROOF. Let S be an infinite convex subset of G, and let n be a positive integer. We must show that $ng\in S$ for some g in G. This follows from Proposition 3 if there exists an infinite interval in S. Suppose that all intervals $(a<b)$ with $a, b\in S$ are finite. In particular, G is discretely ordered. Let C be the cyclic convex subgroup of G that covers 0 and let c be the positive generator of C. An inner automorphism of G preserves order and hence must induce the identity automorphism on C. Therefore C is in the center of G. If $a, b\in S$ and $a<b$, then $a\equiv b \bmod C$. For otherwise $C+a<C+b$ in the divisible group G/C, and it follows that $(a<b)$ is an infinite interval, a contradiction. Thus we have that $S=C^*+s$, where s is a fixed element in S and C^* is an infinite convex subset of C. Since G/C is divisible, $nd\equiv s \bmod C$ for some d in G. Thus $nd=qc+s$ for some integer q, and it is clear that $(nt+q)c\in C^*$ for some integer t. Therefore $n(tc+d)=ntc+nd$ $=ntc+qc+s=(nt+q)c+s\in S$.

COROLLARY. *If Q is a convex subgroup of G that covers* 0, *and if G/Q is divisible then G is regular.*

PROOF. Let S be an infinite convex subset of positive elements in G, and let n be a positive integer. If S contains no infinite interval, then, by the proof of Proposition 4, $ng\in S$ for some $g\in G$. Suppose that $(a<b)$ is an infinite interval in S. If $Q+a<Q+b$, then $Q+a<Q+ng<Q+b$ for some $g\in G$ because G/Q is divisible. Thus $ng\in(a<b)$. If $Q+a=Q+b$, then $nd\equiv a\equiv b \bmod Q$ for some $d\in G$ and by the proof of Proposition 3 (where $C'=Q$ and $C=0$), there exists an element y in Q such that $a<n(y+d)<b$.

3. **The main theorems.** The following two theorems are immediate consequences of the propositions in the last section and their corollaries.

THEOREM I. *Suppose that every convex subgroup of the o-group G is normal.*

(a) *G is regular if and only if G/C is divisible for every nonzero convex subgroup C of G. In particular, if $G\neq 0$ is abelian and regular, then G is divisible if and only if it contains a nonzero divisible convex subgroup.*

Suppose, in addition, that G is discretely ordered and let Q be the convex subgroup of G that covers 0.

(b) *G is $\mathfrak{J}$-regular if and only if G/Q is densely ordered and G/C is divisible for every convex subgroup C of G that properly contains Q.*

THEOREM II. *Suppose that G is an o-group with a convex subgroup Q that covers* 0.

(a) *G is regular if and only if G/Q is divisible. In particular, if G is regular and a central extension of Q, then G is divisible if and only if Q is divisible. Suppose, in addition, that G is discretely ordered ($\equiv Q$ is discretely ordered) and that Q' is a convex subgroup of G that covers Q.*

(b) *G is $\mathfrak{J}$-regular if and only if G/Q is densely ordered and G/Q' is divisible.*

Note that Theorem II characterizes regular and $\mathfrak{J}$-regular groups of finite rank. Also, a regular discretely ordered abelian group is an extension of an infinite cyclic group by a rational vector space.

4. Examples and remarks. Let A be the direct sum of the groups D of rational numbers and the integral multiples of π with the natural order. Let I be the group of integers, and let $G=I\oplus A$ lexicographically ordered from the right. Thus G/I is densely ordered, but not divisible. By Theorem II, G is $\mathfrak{J}$-regular but not regular.

For each $m=-1, -2, \cdots$, let D_m be the group of rationals. Let A be the large direct sum $\cdots\oplus D_{-2}\oplus D_{-1}$, lexicographically ordered from the right, and let B be the subgroup of A that is generated by the small direct sum of the D_m and the "long integral constants"

$$(\cdots, n, n, n)$$

where n is an integer. Let $G=I\oplus B$ lexicographically ordered from the right. G/I is not divisible, and hence by Theorem II, G is not regular, but G/I is densely ordered and G/C is divisible for all convex subgroups C of G that properly contain I. Thus, by Theorem I, G is $\mathfrak{J}$-regular. Note that B is regular, and densely ordered, but not divisible.

We next give an example of an o-group G such that G/C is divisible for all nonzero convex normal subgroups C of G, but G is not $\mathfrak{J}$-regular. This helps to justify the hypothesis in Theorem I that all convex subgroups are normal. Let G be the wreath product of the integers I by the rationals D. Then G is a splitting extension of the small direct sum S of D copies of the integers by D. If we order G lexicographically, then S is the only proper convex normal subgroup of G, and G/S is divisible, because it is isomorphic to D. If C is any convex subgroup of S, then S/C is not divisible, hence S is not regular, and thus G is not regular. Since G is densely ordered, G is also not $\mathfrak{I}$-regular.

Let us call an o-group G $\mathfrak{N}$-regular if for every nonempty interval $(a<b)$ in G and every positive integer n, there exists an element g in G such that $ng\in(a<b)$. Clearly if G is $\mathfrak{N}$-regular, then it is densely ordered, and hence each nonempty interval is infinite. Therefore G is $\mathfrak{N}$-regular if and only if G is regular and densely ordered.

Let A be an abelian o-group with a convex subgroup Q that covers 0. For each positive integer n, the mapping α_n of $nQ+q$ upon $nA+q$ is an isomorphism of Q/nQ into A/nA. The following are equivalent:

(i) A is regular,

(ii) A/Q is divisible,

(iii) α_n is an isomorphism of Q/nQ *onto* A/nA for each $n>0$.

Proof. (i) and (ii) are equivalent by Theorem II. Clearly (iii) is equivalent to $a\equiv x \bmod nA$ having a solution in Q for each $a\in A$ and each $n>0$. Consider $a\in A$ and $n>0$. If (iii) is satisfied, then $a=q+na'$ for some $q\in Q$ and $a'\in A$, and hence $Q+a=Q+na'=n(Q+a')$. Thus A/Q is divisible. Conversely, if A/Q is divisible, then $Q+a=Q+na'$ for some $a'\in A$, hence $a\equiv x \bmod nA$ has a solution in Q. Thus (iii) is satisfied.

In particular, if A is discretely ordered, then Q/nQ is a cyclic group of order n. Thus a discretely ordered abelian group A is regular if and only if A/nA is of order n for all $n>0$. This is Zakon's Theorem 2.1. Similarly other results of Zakon can be proven using Theorems I and II. For example, suppose that A_0 and A are regular discretely ordered abelian groups and that $Q\subseteq A_0\subseteq A$, where Q is the convex subgroup of A that covers 0. Then, by Theorem II, A/Q and A_0/Q are divisible (and torsion free), hence $A/Q=A_0/Q\oplus D$, where D is divisible and torsion free. Thus $D\cong(A/Q)/(A_0/Q)\cong A/A_0$ is divisible and torsion free. Therefore A_0 is pure and basic in A. This is Zakon's Corollary 1.2.

References

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