A NOTE ON THE GREATEST CROSSNORM

A. F. RUSTON

Schatten has shown [5, Lemma 2, p. 323; 6, Lemma 3.7, p. 55] that, if $\mathcal{M}$ is a closed subspace of a Banach space $\mathcal{B}$, and there is a projection of $\mathcal{B}$ onto $\mathcal{M}$ with bound unity, then the greatest cross-norm on the tensor product $\mathcal{B} \otimes \mathcal{M}$ is an extension of the greatest cross-norm on $\mathcal{M} \otimes \mathcal{N}$ for any Banach space $\mathcal{N}$.

Now it is known that there is a projection with bound unity of the second conjugate $\mathcal{B}^{**}$ of a Banach space $\mathcal{B}$ onto $\mathcal{B}_0$ (the canonical image of $\mathcal{B}$ in $\mathcal{B}^{**}$) for conjugate spaces $\mathcal{B}$ and for some others [3, p. 580], though not for all Banach spaces (cf. [7]). For such spaces, then, the greatest crossnorm on $\mathcal{B}^{**} \otimes \mathcal{N}$ is an extension of the greatest crossnorm on $\mathcal{B}_0 \otimes \mathcal{N}$. The purpose of this note is to show that the restriction to such spaces is unnecessary. (N.B. $\mathcal{B}$ is sometimes embedded in $\mathcal{B}^{**}$ by identifying it with $\mathcal{B}_0$.)

**Theorem.** Let $\mathcal{B}$ and $\mathcal{N}$ be any Banach spaces. Then the greatest crossnorm on $\mathcal{B}^{**} \otimes \mathcal{N}$ is an extension of the greatest crossnorm on $\mathcal{B}_0 \otimes \mathcal{N}$ (where $\mathcal{B}_0$ is the canonical image of $\mathcal{B}$ in $\mathcal{B}^{**}$).

Let $\mathcal{X}$ be any element of $\mathcal{B}_0 \otimes \mathcal{N} \subseteq \mathcal{B}^{**} \otimes \mathcal{N}$. Clearly (in the notation of [2, §2.4, pp. 347–351])

$$\gamma\{\mathcal{B}^{**} \otimes \mathcal{N}\}(\mathcal{X}) \leq \gamma\{\mathcal{B}_0 \otimes \mathcal{N}\}(\mathcal{X})$$

(since the infimum on the left-hand side is taken over a larger collection of expressions). On the other hand, there exists a continuous

References


University of California, Davis

Received by the editors October 5, 1961.
linear functional $\mathcal{F}$ over $\mathfrak{B}_0 \odot \mathfrak{R}$ with $\mathcal{F}(\mathfrak{x}) = \gamma\{\mathfrak{B}_0 \odot \mathfrak{R}\}(\mathfrak{x})$ and $\|\mathcal{F}\| = 1$ [1, Theorem 2.9.3, p. 19]. Now $\mathcal{F}$ can be associated (cf. [4, Theorem 1.2, p. 78; 6, Theorem 3.2, p. 47]) with a continuous linear operator $T$ on $\mathfrak{R}$ into $\mathfrak{B}^*$ with the same norm as $\mathcal{F}$ by the rule

$$\mathcal{F}(\mathfrak{x} \otimes \mathfrak{y}) = (Ty)(x) \quad (x \in \mathfrak{B}, \mathfrak{y} \in \mathfrak{R}),$$

where $\mathfrak{x}$ is the canonical image of $x$ in $\mathfrak{B}^{**}$. We now construct a continuous linear operator $T'$ on $\mathfrak{R}$ into $\mathfrak{B}^{***}$ by defining $T'y$ to be the canonical image of $Ty$ in $\mathfrak{B}^{***}$ for each $y$ of $\mathfrak{R}$. This is associated with a continuous linear functional $\mathcal{F}'$ over $\mathfrak{B}^{**} \odot \mathfrak{R}$ with the same norm as $T'$ by the rule

$$\mathcal{F}'(X \otimes \mathfrak{y}) = (T'y)(X) \quad (X \in \mathfrak{B}^{**}, \mathfrak{y} \in \mathfrak{R}).$$

Then

$$\mathcal{F}'(\mathfrak{x} \otimes \mathfrak{y}) = (T'y)(\mathfrak{x}) = \mathfrak{x}(Ty) = (Ty)(x) = \mathcal{F}(\mathfrak{x} \otimes \mathfrak{y}),$$

and so $\mathcal{F}'$ is an extension of $\mathcal{F}$, and

$$\|\mathcal{F}'\| = \|T'\| = \|T\| = \|\mathcal{F}\| = 1.$$  

Hence

$$\gamma\{\mathfrak{B}_0 \odot \mathfrak{R}\}(\mathfrak{x}) = |\mathcal{F}(\mathfrak{x})| = |\mathcal{F}'(\mathfrak{x})| \leq \gamma\{\mathfrak{B}^{**} \odot \mathfrak{R}\}(\mathfrak{x}).$$

This inequality, in conjunction with that above, shows that

$$\gamma\{\mathfrak{B}^{**} \odot \mathfrak{R}\}(\mathfrak{x}) = \gamma\{\mathfrak{B}_0 \odot \mathfrak{R}\}(\mathfrak{x}).$$

Since the element $\mathfrak{x}$ of $\mathfrak{B}_0 \odot \mathfrak{R}$ was arbitrary, this completes the proof of the theorem.

References

5. ———, On projections with bound 1, Ann. of Math. (2) 48 (1947), 321–325.
7. A. Sobczyk, Projection of the space $(m)$ on its subspace $(c_0)$, Bull. Amer. Math. Soc. 47 (1941), 938–947.

University of Sheffield, Sheffield, England