$W$ is $n$-parallelisable. Now by the main theorem of [2], $\partial W$ bounds a contractible manifold, and so represents the zero element of $\Gamma_2$.  

**References**


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**THE COEFFICIENTS IN THE EXPANSION OF CERTAIN PRODUCTS**

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1. The identities

$$\prod_{n=0}^{\infty} (1 - p^nx)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{(1 - p)(1 - p^2) \cdots (1 - p^n)},$$

$$\prod_{n=0}^{\infty} (1 - p^n) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{n(n-1)/2} x^n}{(1 - p)(1 - p^2) \cdots (1 - p^n)},$$

where $|p| < 1$, are well known. The more general products

$$\prod_{m,n=0}^{\infty} (1 - p^m g^n x)^{-1}, \quad \prod_{m,n=0}^{\infty} (1 - p^m g^n x) \quad (|p| < 1, |q| < 1)$$

have been discussed in [1, 2].

In the present note we consider the products

$$\prod_{n=0}^{\infty} (1 - p^n x - q^n y)^{-1}, \quad \prod_{n=0}^{\infty} (1 - p^n x - p^n y) \quad (|p| < 1, |q| < 1).$$

Put

$$F(x, y) = \prod_{n=0}^{\infty} (1 - p^nx - q^n y)^{-1} = \sum_{r,s=0}^{\infty} A_{rs} x^r y^s,$$

where $A_{rs} = A_{rs}(p, q)$ is independent of $x$ and $y$. It follows from (4) that

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**Presented to the Society, November 10, 1961; received by the editors November 7, 1961.**

**1 Supported in part by National Science Foundation grant G-16485.**
(1 - x - y)F(x, y) = F(px, qy),

so that

\[(1 - p^s q^s) A_{rs} = A_{r-1,s} + A_{r,s-1} \quad (r + s > 0).\]

Making use of (5) we get for the first few values of $A_{rs}$:

\[
A_{00} = 1, \quad A_{10} = \frac{1}{1 - p}, \quad A_{01} = \frac{1}{1 - q},
\]

\[
A_{20} = \frac{1}{(1 - p)(1 - p^2)}, \quad A_{02} = \frac{1}{(1 - q)(1 - q^2)},
\]

\[
A_{11} = \frac{1}{(1 - p)(1 - pq)} + \frac{1}{(1 - q)(1 - pq)}.
\]

It is evident from (1) that

\[
A_{r0} = \frac{1}{(1 - p)(1 - p^2) \cdots (1 - p^r)},
\]

\[
A_{0r} = \frac{1}{(1 - q)(1 - q^2) \cdots (1 - q^r)}.
\]

Also it is clear from (3) that

\[
A_{rs}(p, q) = A_{sr}(q, p).
\]

In (5) take $s=1$, so that

\[
(1 - p^s q^s) A_{r1} = A_{r-1,1} + A_{r,0}.
\]

Making use of (6) and (8) we find that

\[
A_{r1} = \sum_{j=0}^{r} \frac{1}{(1 - p) \cdots (1 - p^j)(1 - p^j q) \cdots (1 - p^r q)}.
\]

We next take $s=2$ in (5) and combine with (9) to get

\[
A_{r2} = \sum_{0 \leq j_1 \leq \cdots \leq j_s \leq r} \frac{1}{(1 - p) \cdots (1 - p^{j_1})(1 - p^{j_1} q) \cdots (1 - p^{j_s} q)(1 - p^{j_s} q^s) \cdots (1 - p^{j_s} q^r)}.
\]

It is now not difficult to state the general result, namely

\[
A_{rs} = \sum_{0 \leq j_1 \leq \cdots \leq j_s \leq r} \frac{1}{(1 - p) \cdots (1 - p^{j_1})(1 - p^{j_1} q) \cdots (1 - p^{j_s} q)(1 - p^{j_s} q^s) \cdots (1 - p^{j_s} q^r)},
\]

where the summation is over all $j_1, j_2, \ldots, j_s$ such that

\[
0 \leq j_1 \leq j_2 \leq \cdots \leq j_s \leq r.
\]
The proof of (11) is by induction on \( s \) and will be omitted.

As a partial verification of (11) we note that since the number of solutions of (12) is equal to

\[
\binom{r + s}{r} = \frac{(r + s)!}{r!s!},
\]

it follows that when \( p = q \), (11) reduces to

\[
A_{rs}(p, q) = \binom{r + s}{r} \frac{1}{(1 - p) \cdots (1 - p^{r+s})}
\]

which agrees with (1).

2. Turning next to the second product in (3) we put

\[
G(x, y) = \prod_{n=1}^{\infty} (1 - p^n x - q^n y) = \sum_{r,s=0}^{\infty} B_{rs} x^r y^s.
\]

It follows from (13) that

\[
(1 - px - qy)G(px, qy) = G(x, y),
\]

so that

\[
(1 - p^{r}q^{s})B_{rs} = -p^{r}q^{s}(B_{r-1,s} + B_{r,s-1}).
\]

Also comparing (13) with (2) we have

\[
B_{r0} = \frac{(-1)^{r}p^{r(r+1)/2}}{(1 - p) \cdots (1 - p^{r})}.
\]

Now put

\[
B_{rs}^* = B_{rs}(p, q) = B_{rs}\left(\frac{1}{p}, \frac{1}{q}\right).
\]

Then (14) becomes

\[
(1 - p^{r}q^{s})B_{rs}^* = B_{r-1,s}^* + B_{r,s-1}^*.
\]

Since by (15)

\[
B_{r0}^* = \frac{1}{(1 - p) \cdots (1 - p^{r})} = A_{r0},
\]

comparison of (16) with (5) yields

\[
B_{rs}^* = A_{rs}.
\]
Therefore we have

\[ B_{ns}(p, q) = A_{rs} \left( \frac{1}{p}, \frac{1}{q} \right) \]

and \( B_{ns} \) is determined explicitly by means of (11).

If we let \( A_{ns}(j_1, \ldots, j_s) \) denote the summand in the right member of (11) and \( A_{ns}^+(j_1, \ldots, j_s) \) the corresponding function with \( p, q \) replaced by \( p^{-1}, q^{-1} \), respectively, it follows that

\[ A_{ns}^+(j_1, \ldots, j_s) = (-1)^{s+r} p^{r(s+1)/2+s+r+\cdots+j_s} q^{s(s+1)/2+s+\cdots+j_s} A_{ns}(j_1, \ldots, j_s). \]

Note that when \( p = q \) the sum of the exponents on \( p \) and \( q \) is equal to

\[ \frac{1}{2} r(r+1) + \frac{1}{2} s(s+1) + rs = \frac{1}{2} (r+s)(r+s+1), \]

which is correct.

In terms of \( A_{ns}(j_1, \ldots, j_s) \) and \( A_{ns}^+(j_1, \ldots, j_s) \) we have

\[ A_{rs} = \sum A_{ns}(j_1, \ldots, j_s), \]

\[ B_{ns} = \sum A_{ns}^+(j_1, \ldots, j_s), \]

where in each case the summation is over all \( j_1, \ldots, j_s \) satisfying (12).

From the definition of \( A_{ns}(j_1, \ldots, j_s) \) we have

\[ A_{ns}(j_1, \ldots, j_s) = \frac{A_{r+s-1}(j_1, \ldots, j_{s-1})}{(1 - p^r q^s) \cdots (1 - p^r q^s)}; \]

therefore (20) yields

\[ A_{rs} = \sum_{j=0}^{r} \frac{A_{j,s-1}}{(1 - p^j q^s) \cdots (1 - p^j q^s)}, \]

which can also be obtained from (5).

We remark that the pair of formulas

\[ (r + 1) A_{r+1,s} = \sum_{j=0}^{r} \sum_{k=0}^{s} \binom{j+k}{j} A_{r-j,s-k} \frac{1}{1 - p^{j+1} q^k}, \]

\[ (s + 1) A_{r,s+1} = \sum_{j=0}^{r} \sum_{k=0}^{s} \binom{j+k}{j} A_{r-j,s-k} \frac{1}{1 - p^j q^{k+1}} \]

can be proved by logarithmic differentiation of (4).

3. We shall now determine the coefficients in the expansion
A triple $T$ is an ordered set of three non-negative integers $i, j, k$. Each of the triples $(i - 1, j, k), (i, j - 1, k), (i, j, k - 1)$ precedes $i, j, k$; notation $T_1 < T$. A chain $C$ is a set of triples:

$$T_1 < T_2 < \cdots < T_k,$$

where $T_1 = (1, 0, 0), (0, 1, 0)$ or $(0, 0, 1); T_k$ is the last element of $C$. Corresponding to the triple $i, j, k$, we put

$$(26) \quad \pi(C) = \prod_{T \in C} (1 - p_i^j p_j^k),$$

where $1 - p_i^j p_j^k$ corresponds to the triple $T = (i, j, k)$.

We shall show that

$$(27) \quad A_{rt} = \sum_c \frac{1}{\pi(C)},$$

where the summation is over all chains with last element $(r, s, t)$.

In the first place it is clear from (25) that

$$(28) \quad (1 - p_1 p_2 p_3) A_{rt} = A_{r-1,s,t} + A_{r,s-1,t} + A_{r,s,t-1}$$

and

$$(29) \quad A_{rt0} = A_{s}(p_1, p_2), \quad A_{rt} = A_{s}(p_1, p_3), \quad A_{0st} = A_{s}(p_2, p_3),$$

where $A_{s}(p_1, p_2)$ is defined by (4). Moreover $A_{rt}$ is uniquely determined by means of (28) and (29).

Now if $A_{rt}$ is defined by (27) it follows from (11) and (27) that (28) is satisfied. To show that (28) is also satisfied we remark that if $C$ is a chain with last element $(r, s, t)$, then deleting this element we are left with a chain whose last element is $(r - 1, s, t), (r, s - 1, t)$ or $(r, s, t - 1)$ and conversely. In view of (26), (28) follows immediately.

If we put

$$(30) \quad \prod_{n=0}^{\infty} (1 - p_1 x - p_2 y - p_3 z) = \sum_{r,s,t=0}^{\infty} B_{rst} x^r y^s z^t$$

then, exactly as in the proof of (18), we have

$$(31) \quad B_{rst} = A_{rst} \left( \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \right).$$

It is clear from (26), (27) and (31) how the coefficients in the expansion of
\[ \prod_{n=0}^{\infty} (1 - p_1x_1 - \cdots - p_kx_k)^{-1} \]

and

\[ \prod_{n=1}^{\infty} (1 - p_1x_1 + \cdots + p_kx_k) \]

can be determined for all \( k \geq 1 \).

Added in proof. Professor B. M. Bennett has kindly informed the writer that he has obtained a formula equivalent to (11) above in his paper: On a rank-order test for the equality of probabilities in multinomial trials. Also it is evident from his paper that the coefficients \( A_{n}(p, q) \) are of some statistical interest.

References


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