1. Introduction. Let $A$ be a simple, flexible, powerassociative, finite-dimensional algebra over a field of characteristic zero. Then it is known that $A$ has a unity element $1$ [5], and consequently $A$ has a degree. When $A$ has degree larger than two, Oehmke has shown [5] that $A^+$ is a simple Jordan algebra. Kokoris [4] has shown the same result in case $A$ has degree two. In this paper we are able to show that if $A$ has degree one then in fact $A$ must be a one-dimensional algebra. Combining these results, the following theorem may be asserted.

Main Theorem. If $A$ is a simple, flexible, powerassociative, finite-dimensional algebra of characteristic zero then $A^+$ is a simple Jordan algebra.

2. Proof. We begin with a result that is more general than actually needed to prove the main theorem.

Theorem 1. Let $R$ be a flexible algebra with unity element $1$ over a field $F$ of characteristic not two. Suppose there exists some vector space decomposition of $R$, $R = F1 + N$, such that for all elements $a, b$ in $N$ $a \cdot b = (ab + ba)/2$ is in $N$. Then the ideal $C$ generated by all elements of the form $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is contained in $N$ and hence is a proper ideal of $R$.

Proof. For arbitrary elements $x_1, x_2, y$ in $N$ we have $x_1y = \lambda_11 + z_1$, and $x_2y = \lambda_21 + z_2$, where $z_1$ and $z_2$ are in $N$, while $\lambda_1, \lambda_2$ are scalars. As in Schafer [7, Relation (8)] it follows from the flexible law that

$$
(x_1, x_2)y = \lambda_1x_2 + \lambda_2x_1 + x_1 \cdot z_2 + x_2 \cdot z_1 - (x_1 \cdot y) \cdot x_2 - (x_2 \cdot y) \cdot x_1
$$

(1)

$$
+ (x_1 \cdot x_2) \cdot y.
$$

As in Kokoris [3, p. 653] one goes on to show from (1) that

$$
(x_1, x_2, x_3)y = (x_1, x_2, z_2) + (x_1, z_2, x_3) + (z_1, x_2, x_3) - (x_1 \cdot y, x_2, x_3)
$$

(2)

$$
- (x_1, x_2 \cdot y, x_3) + (x_3 \cdot y, x_2, x_1) + (x_1, x_2, x_3) \cdot y,
$$

where $(x, y, z)$ is defined here as $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$, while $x_3y = \lambda_31 + z_3$, where $z_3$ is in $N$ and $\lambda_3$ is a scalar. Then if $B$ is the subspace generated by all $(x, y, z)$, relation (2) implies that $BN \subset B$.

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+B \cdot N, \quad NB \subset B + B \cdot N, \quad \text{and more generally} \quad ((BN) \cdots) N \subset B + B \cdot N + \cdots + ((B \cdot N) \cdots) \cdot N \text{ etc. As a result the set } C, \text{ defined as the set of all finite sums of elements from the sets } B, \quad B \cdot N, \quad (B \cdot N) \cdot N, \quad \text{etc., can be shown to be an ideal of } A. \text{ Since } B \text{ is readily shown to be in } N \text{ and since } N \cdot N \subset N \text{ by hypothesis, we may conclude that } C \subset N. \text{ This concludes the proof of the theorem.}

**Corollary.** If \( R \) is also assumed to be simple then \( R^+ \) is an associative, commutative algebra.

While the following theorem is not essential to the proof of the Main Theorem, it together with Theorem 1 might be useful in a study of flexible algebras where the elements of \( N \) are not necessarily nilpotent.

**Theorem 2.** If \( S \) is a flexible ring of characteristic different from two such that \( S^+ \) is power associative, then \( S \) must be power associative.

**Proof.** From the flexible law third-power associativity follows. Assume inductively \( k \)-power associativity for all \( k < n \). We proceed to establish \( w \)-power associativity. The flexible law implies that

\[
x^{a-1}x = (xx^{a-1})x = x(x^{a-1}) = xx^{a-1}.
\]

By a second induction suppose \( x^{a-2}x^a = x^ax^{a-2} \) for \( 0 < a < n - 1 \). We have already established this for \( a = 1 \). The linearized form of the flexible law implies that

\[
(x^{a-1}x)x^a + (x^ax)x^{a-1} = x^{a-1}(xx^a) + x^a(xx^{a-1}).
\]

By the second induction hypothesis the first term on the left cancels the second term on the right in the last equality, leaving

\[
x^{n-(a+1)}x^{a+1} = x^{(a+1)}x^{n-(a+1)}.
\]

This completes the proof of the second induction. Powerassociativity in \( A^+ \) implies

\[
(x^a \cdot x^{n-a-1}) \cdot x = x^a \cdot (x^{n-a-1} \cdot x).
\]

However from this it follows that

\[
2x^{n-1} \cdot x = 2x^{n-1} \cdot x^a,
\]

so that, for all \( a \), \( x^{n-1} \cdot x = x^{n-a} \cdot x^a \), assuming characteristic different from two. This completes the first induction and the proof of the theorem.

We note that in general powerassociativity of \( T^+ \) does not suffice to guarantee powerassociativity of \( T \).
Corollary. If R is simple then R must be powerassociative.

Consider now the case at hand, in which A is assumed to have degree one over an algebraically closed field. Then there exists a vector space decomposition \( A = F1 + N \), where in fact all elements of \( N \) are nilpotent. Albert [2, p. 527] has shown that in \( A^+ \), \( N \) is a subalgebra. From this one infers that \( A \) satisfies the hypotheses of Theorem 1. From the Corollary to Theorem 1 it follows that \( A^+ \) is associative. Hence \( A \) is a noncommutative Jordan algebra. At this point a result of Schafer's [6, Main Theorem] may be used to conclude that \( A \) is trace-admissible. Albert [1, Principal Theorem] has shown that a trace-admissible algebra \( A \) is simple if and only if \( A^+ \) is simple. Thus \( A^+ \) is a simple, associative, commutative, finite-dimensional algebra. Then it is well known that \( A^+ \) must be a field. Therefore \( N \) must be zero. This of course means \( A \) is isomorphic to \( F \). We have proved

Theorem 3. If \( A \) is a simple, flexible, powerassociative, finite-dimensional algebra over an algebraically closed field of characteristic zero and degree one then \( A \) is a one dimensional field.

The existence of nodal, noncommutative Jordan algebras indicates that the conclusion of Theorem 3 is not true for fields of finite characteristics [3].

References


Ohio State University and
Illinois Institute of Technology