THE RADIUS OF GYRATION OF A CONVEX BODY

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The purpose of this note is to establish an inequality of isoperimetric type for convex bodies. Let \( K \) be a bounded convex body and \( l \) a line through its centroid; let \( \delta(K, l) \) be the supremum of distances of points of \( K \) from \( l \), \( I(K, l) \) the moment of inertia of \( K \) about \( l \), and \( g(K, l) \) the radius of gyration \( [I(K, l)/\text{mass } K]^{1/2} \) of \( K \) about \( l \). If \( K \) were not restricted to be convex, it is easy to show that the values of the ratio \( g(K, l)/\delta(K, l) \) consist of all numbers in the open interval \((0, 1)\). However, under the restriction to convex \( K \) we shall show that the infimum of this ratio is \( 1/15^{1/2} \). The usual methods of the calculus of variations are unavailable, as is usual in problems concerning convex bodies.

If \( K \) is a bounded convex body in three-space, and \( l \) is a line in three-space, and \( \delta(x, y, z; l) \) the distance from the point \((x, y, z)\) to \( l \), we define the mass, \( m(K) \), centroid \((\bar{x}(K), \bar{y}(K), \bar{z}(K))\), moment of inertia \( I(K, l) \) and radius of gyration \( g(K, l) \) as usual:

\[
m(K) = \int_K dx \, dy \, dz, \quad \bar{x}(K) = \int_K x \, dx \, dy \, dz/m(K), \text{ etc.,}
\]

\[
I(K, l) = \int_K \delta(x, y, z; l)^2 \, dx \, dy \, dz, \quad g(K, l) = [I(K, l)/m(K)]^{1/2};
\]

and we define

\[
\delta(K, l) = \sup \{\delta(x, y, z; l) : (x, y, z) \in K\}.
\]

It is obvious that none of these quantities change if we replace \( K \) by its closure. Likewise, if a linear mass-distribution in an interval \([a, b]\) is defined by a density \( A(x) \) \((a \leq x \leq b)\), and \( c \) is a real number, we define

\[
m(A) = \int_a^b A(x) \, dx, \quad \bar{x}(A) = \int_a^b x \, A(x) \, dx/m(A),
\]

\[
I(A, c) = \int_a^b (x - c)^2 \, A(x) \, dx, \quad g(A, c) = [I(A, c)/m(A)]^{1/2},
\]

\[
\delta(A, c) = \text{larger of } |a - c|, \quad |b - c|.
\]

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We simplify the notation when \( c = \delta(A) \), writing

\[ I(A) = I(A, \delta(A)), \quad g(A) = g(A, \delta(A)), \quad \delta(A) = \delta(A, \delta(A)). \]

Theorem. If \( K \) is a bounded convex body and \( l \) a line through its centroid, and \( \delta(K, l) \) is the supremum of distances of points of \( K \) from \( l \) and \( g(K, l) \) the radius of gyration of \( K \) about \( l \), then

\[ g(K, l)/\delta(K, l) > 1/15^{1/2}. \]

In this statement the constant \( 1/15^{1/2} \) cannot be replaced by any larger number.

By the remark after (2), we may and shall restrict our attention to closed convex bodies. Given any \( K \) and \( l \) as in the theorem, we choose rectangular axes so that \( l \) is the \( z \)-axis and the positive \( x \)-axis contains a point at distance \( \delta(K, l) \) from \( l \). The projection of \( K \) on the \( x \)-axis is an interval \([a, b]\) with \(-\delta(K, l) \leq a < b = \delta(K, l)\). For each \( x \) in \([a, b]\) let \( A(x) \) be the area of the intersection \( K(x) \) of \( K \) with the plane \( x = \xi \). By the Brunn-Minkowski theorem [1, p. 88] \( [A(x)]^{1/2} (a \leq x \leq b) \) is a concave function, obviously continuous in the open interval \((a, b)\) and by the closure of \( K \) continuous at \( a \) and \( b \) also. Also, \( \delta(A) = \delta(K) = 0 \), so \( \delta(A) = b = \delta(K, l) \). For the moments of inertia we have

\[
I(K, l) = \int_a^b \left\{ \int_{K(x)} (x^2 + y^2) dy \right\} dz \, dx
\]

\[
> \int_a^b \int_{K(x)} x^2 dy \, dz \, dx
\]

\[
= \int_a^b x^2 A(x) dx = I(A),
\]

so

\[ g(A)/\delta(A) < g(K, l)/\delta(K, l). \]

Let \( \mathcal{A} \) be the family of functions \( A \) each defined, continuous and non-negative on some closed interval and having \( [A]^{1/2} \) concave on that interval. The function \( A \) of the preceding paragraph belongs to \( \mathcal{A} \). So if we define

\[ \mu = \inf \{ g(K, l)/\delta(K, l) \} \text{ for all convex bodies } K \text{ and all lines through the centroid of } K, \]

\[ \mu' = \inf \{ g(A)/\delta(A) \} \text{ for all } A \text{ in } \mathcal{A}, \]

by (6) we have
(8) \[ \mu' \leq \mu. \]

On the other hand, let \( A(x)(a \leq x \leq b) \) belong to \( \mathcal{F} \). For each positive \( \varepsilon \) let \( K_\varepsilon \) be the solid of revolution obtained by revolving the set \( \{(x, y) : a \leq x \leq b, 0 \leq y \leq \varepsilon [A(x)/\pi]^{1/2}\} \) about the x-axis. Then \( K_\varepsilon \) is convex, and its centroid is \((\bar{x}(A), 0, 0)\). Let \( l \) be the line \( x = \bar{x}(A), y = 0 \). It is easy to see that as \( \varepsilon \) tends to 0, \( \delta(K_\varepsilon, l) \) tends to \( \delta(A) \). Also,

\[ I(K_\varepsilon, l) = \int_a^b \left[ (x - \bar{x}(A))^2 + \varepsilon^2 A(x)/4\pi \right] \varepsilon^2 A(x) dx, \]

\[ m(K_\varepsilon) = \int_a^b \varepsilon^2 A(x) dx, \]

so

\[ \lim_{\varepsilon \to 0} \frac{g(K_\varepsilon, l)}{\delta(K_\varepsilon, l)} = \left[ \frac{\int_a^b (x - \bar{x}(A))^2 A(x) dx}{\int_a^b A(x) dx} \right]^{1/2} = g(A). \]

Therefore as \( \varepsilon \) tends to 0, \( g(K_\varepsilon, l)/\delta(K_\varepsilon, l) \) tends to \( g(A)/\delta(A) \), and \( \mu \) cannot be greater than \( \mu' \). This and (8) prove that

(9) \[ \mu = \mu', \]

and (9) and (6) prove that for all \( K \) and \( l \) as in the theorem, \( g(K, l)/\delta(K, l) > \mu \). It remains to prove that \( \mu = 1/15^{1/2} \). The function \( A(x) = (1 - x)^2 (0 \leq x \leq 1) \) belongs to \( \mathcal{F} \), and we readily compute that for it

\[ g(A)/\delta(A) = 1/\sqrt{15}. \]

Hence

(10) \[ \mu \leq 1/\sqrt{15}. \]

Suppose now that \( \mu < 1/15^{1/2} \). Let \( A_1, A_2, \ldots \) be a sequence of functions belonging to \( \mathcal{F} \) such that

(11) \[ \lim_{n \to \infty} \frac{g(A_n)}{\delta(A_n)} = \mu. \]

The ratio \( g(A)/\delta(A) \) is unaltered by transformations \( x \to hx + k \) with \( h > 0 \), so we may suppose that all the \( A_n \) have \([0, 1]\) as domain. Also, the ratio is unaffected by multiplication of \( A \) by a positive constant, so we may suppose that all \( A_n \) have supremum 1. Then for each positive integer \( k \) all the functions \( A_n \) satisfy the Lipschitz condition with constant \( 1/k \) on the interval \([1/k, 1 - 1/k] \), so we may select a subsequence uniformly convergent on that interval. By the diagonal process we obtain a subsequence converging to a limit \( A(x) \) for each \( x \) in \((0, 1)\). Then \( [A]^{1/2} \) is concave, so \( A(x) \) has limits as \( x \to 0+ \) and
as \( x \to 1 - \). We assign \( A(0) \) and \( A(1) \) these respective values. Then \( A \) belongs to \( \mathcal{F} \) and satisfies

\[
\frac{g(A)}{\delta(A)} = \mu.
\]

Since multiplication by a positive constant leaves \( A \) in \( \mathcal{F} \) and leaves the left member of (12) unchanged, we may assume \( m(A) = 1 \). Likewise, by applying the transformation \( x \to 1 - x \) if necessary, we may obtain \( \bar{x}(A) \leq 1/2 \). From \( A \) we construct \( K \), by rotation of \( \epsilon[A/\pi]^{1/2} \), as before; this is convex and its centroid is \( (\bar{x}(A), 0, 0) \), so [1, p. 52]

\[
1/4 \leq \bar{x}(A) \leq 1/2.
\]

To simplify notation we shall define

\[
c = \bar{x}(A).
\]

Let \( \epsilon \) be any positive number less than 1; the \( O \) and \( o \) notation will be used for estimates with \( \epsilon \) near 0. Define

\[
\alpha = \int_0^1 A(x) \, dx;
\]

then \( \alpha = 0(\epsilon) \). Let \( A_\epsilon \) be the restriction of \( A \) to the subinterval \([\epsilon, 1]\). Then

\[
m(A_\epsilon) = 1 - \alpha,
\]

\[
m(A_\epsilon) \bar{x}(A_\epsilon) = \int_0^1 xA(x) \, dx = c + \alpha O(\epsilon),
\]

whence

\[
\bar{x}(A_\epsilon) = c(1 + \alpha) + \alpha O(\epsilon).
\]

Also

\[
I(A_\epsilon) = I(A_\epsilon, c) = [\bar{x}(A_\epsilon) - c]^2m(A_\epsilon)
\]

\[
= I(A) - \int_0^1 (c - x)^2A(x) \, dx + O(\alpha^2)
\]

\[
= I(A) - c^2\alpha[1 + O(\epsilon)],
\]

whence

\[
g(A_\epsilon)^2 = I(A)\left[1 + \alpha\left[1 - c^2/I(A) + O(\epsilon)\right]\right].
\]

Since

\[
\delta(A_\epsilon) \geq 1 - \bar{x}(A_\epsilon) = \delta(A) - c\alpha + \alpha O(\epsilon),
\]

we have
\[ \left[ \frac{g(A)}{\delta(A)} \right]^2 \leq \frac{I(A)}{\delta(A)^2} \left\{ 1 + \alpha \left[ 1 - \frac{c^2}{I(A)} + 2c/\delta(A) + O(\varepsilon) \right] \right\}. \]

By definition of \( \alpha \), with (12),
\[ I(A)/\delta(A)^2 = \left[ \frac{g(A)}{\delta(A)} \right]^2 = \mu^2 < 1/15, \]
so, (allowing the next equation to define \( C \))
\[ C = 1 - c^2/I(A) + 2c/\delta(A) = 1 - \frac{c^2}{\mu^2 \delta(A)^2} + 2c/\delta(A) \]
\[ = - \left( \frac{c}{\mu \delta(A)} - \mu \right)^2 + \mu^2 + 1. \]

Since \( c/\delta(A) = c/(1-c) \), which is increasing on \([0, 1)\), and \( 1/4 \leq c \leq 1/2 \), we have \( c/\delta(A) \geq 1/3 \). Since \( \mu^2 < 1/15 \), we also have \( c/\mu \delta(A) - \mu > 0 \), and the right member of (17) is not decreased if we replace \( c/\delta(A) \) by \( 1/3 \):
\[ C \leq 1 + \mu^2 - \left( \frac{1}{3\mu} - \mu \right)^2. \]

The derivative with respect to \( t \) of the function \( \phi(t) = (1/(3t) - t)^2 \) is
\[ \phi'(t) = -\frac{2}{t} \left( \frac{1}{9t} - t^2 \right), \]
which is negative for \( t \) in \((0, 1/3)\). Hence the right member of (18) is increased if we replace \( \mu \) by the larger number \( 1/15^{1/2} \), so that
\[ C < 1 + 1/15 - (15^{1/2}/3 - 1/15^{1/2})^2 = 0. \]

The coefficient \( C - O(\varepsilon) \) of \( \alpha \) in (15) is therefore negative for all positive \( \varepsilon \) near 0, and \( \alpha \) is positive, so for all such \( \varepsilon \) we have by (15) and (16)
\[ g(A_\varepsilon)/\delta(A_\varepsilon) < \mu. \]

But this contradicts the definition of \( \mu \), and the proof is complete.

If \( K \) is a right circular cylinder with axis \( l \), it is easily computed that \( g(K, l)/\delta(K, l) = 1/2^{1/2} \). I conjecture, but have not been able to complete a proof, that \( 1/2^{1/2} \) is in fact the supremum of the ratio \( g(K, l)/\delta(K, l) \) for all convex bodies \( K \) and all lines \( l \) through the centroid of \( K \).

**Bibliography**


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