Let $s_n$ be a sequence in a $p$-dimensional Euclidean space $E^p$. Let $K_n = K(s_n, s_{n+1}, \cdots)$ be the convex hull of $s_n, s_{n+1}, \cdots$ and $\overline{K}_n$ its closure. The core of $s_n$ is defined as $\bigcap_{n=1}^{\infty} \overline{K}_n$. Knopp's core theorem states that if $A = (a_{ij})$ is an infinite regular matrix with nonnegative elements, then the core of the $A$-transform of $s_n$ is contained in the core of $s_n$. In particular if $s_n$ is bounded, every $A$-limit of a subsequence of $s_n$ is contained in the convex hull of limit points of $s_n$. With certain restrictions on $A$, the converse is also true; i.e., for any element $\xi$ in the convex hull of limit points of $s_n$, there is a subsequence of $s_n$ which is $A$-limitable to $\xi$. The main objective of this paper is to show that for any $\xi$ in the convex hull of limit points of a bounded sequence $s_n$, there is a subsequence of $s_n$ which is $C_1$ and $E_1$-limitable to $\xi$.

The following is Knopp's core theorem in $E^p$.

**Theorem 1.** Let $s_n$ be a sequence in $E^p$ and $A = (a_{ij})$ a regular matrix with $a_{ij} \geq 0$. Let $K_n$ be the convex hull of $s_n, s_{n+1}, \cdots$ and $K_n'$ the convex hull of $s'_n, s'_{n+1}, \cdots$, where $s'_n = \sum_{j=1}^{m} a_n s_j$ is defined for each $n = 1, 2, \cdots$. Then $\bigcap_{n=1}^{\infty} \overline{K}_n' \subset \bigcap_{n=1}^{\infty} \overline{K}_n$.

**Proof.** For each $\epsilon > 0$, define $K_n = \{x \mid \inf_{y \in K_n} \|x - y\| \leq \epsilon\}$. For given $m$ and $\epsilon > 0$, choose $\nu(m, \epsilon)$ so that

$$\left| \sum_{j=1}^{m-1} a_n s_j \right| < \epsilon, \quad \left| \sum_{j=1}^{\infty} a_n s_j - 1 \right| < \epsilon, \quad \left| \sum_{j=1}^{m-1} a_n \right| < \epsilon \quad \text{for } n \geq \nu.$$ 

Now choose an index $k(k(n))$ so that

$$\left| \sum_{j=m+k+1}^{\infty} a_n s_j \right| < \epsilon, \quad \left| \sum_{j=m+k+1}^{\infty} a_n \right| < \epsilon.$$

Then $\left| \sum_{j=m}^{m+k} a_n s_j - 1 \right| < 3\epsilon$. Assume $\epsilon < 1/3$ so that $\sum_{j=m}^{m+k} a_n s_j - 1 \neq 0$. Let $s'_n = \alpha_n + \beta_n + \gamma_n$ where

$$\alpha_n = \sum_{j=1}^{m-1} a_n s_j, \quad \beta_n = \sum_{j=m+k}^{\infty} a_n s_j, \quad \gamma_n = \sum_{j=m}^{m+k} a_n s_j.$$

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Since $x_n \in K_m$ for every $\epsilon$, $s'_n / \gamma_n \in K_{m'}$ for $n \geq \nu$ where $\epsilon' = 2\epsilon / (1 - 3\epsilon)$, $|\gamma_n - 1| < 3\epsilon$.

Now let $\xi \in K'_m$ for every $\nu$. Then $\xi \in K'_n$ for every $\delta > 0$ and every $n$; therefore $\xi = a_1 s'_{n_1} + \cdots + a_s s'_{n_s} + \eta$ where $\|\eta\| < \delta$, $n_i \geq \nu$ for each $i$, $a_i > 0$ and $\sum_{i=1}^s a_i = 1$. Since $s'_n / \gamma_n \in K_{m'}$ for $n \geq \nu$, $\xi = \eta + \sum_{i=1}^s a_i x_n \gamma_n$, where $x_{n_i} \in K_{m'}$ for $n_i \geq \nu$ and $|\gamma_n - 1| < 3\epsilon$. Let $\gamma = \sum_{i=1}^s a_i \gamma_{n_i}$, then $(1/\gamma)(\sum_{i=1}^s a_i \gamma_{n_i} x_{n_i}) \in K_{m'}$; hence $\xi \in K_m$ for every $m$.

It is clear that $\bigcap_{n=1}^\infty K_n$ always contains limit points of $s_n$; hence it contains the convex hull of limit points of $s_n$. Moreover if $s_n$ is bounded $\bigcap_{n=1}^\infty K_n$ is precisely the convex hull of limit points of $s_n$.

To see this let $Q$ be the set of limit points of $s_n$ and $N_\epsilon(x)$ the $\epsilon$-neighborhood of $x$. Then for each $\epsilon > 0$, $\bigcup_{n=0}^\infty N_\epsilon(x)$ contains all but a finite number of $s_n$. Now choose an index $p$ so that $s_n \in \bigcup_{n=0}^\infty N_\epsilon(x)$ for $n \geq p$. Moreover since $\xi \in K_{m'}$ for every $\delta > 0$ and $m$, $\xi = \sum_{i=1}^s a_i s_{n_i} + \eta$ where $a_i \geq 0$, $\sum_{i=1}^s a_i = 1$, $n_i \geq p$ and $\|\eta\| < \delta$. Therefore there are $\xi_1$, $\xi_2$, $\cdots$, $\xi_s$ in $Q$ such that $\xi = \eta + \sum_{i=1}^s a_i \xi_i + \sum_{i=1}^s a_i \epsilon_i$ where $\|\epsilon_i\| \leq \epsilon$; therefore, $\xi \in K_{m+\delta}(Q)$. Hence $\xi \in K(Q)$. The above remark and Knopp’s core theorem give the following corollary.

**Corollary 1.** If $A$ is an infinite regular matrix with nonnegative elements, then every $A$-limit of a bounded sequence is in the convex hull of the limit points of the sequence.

The following is a sufficient condition on a regular matrix so that the converse of Corollary 1 holds.

**Lemma 1.** Let $A$ be a nonnegative regular matrix. Suppose there is a set of sequences $C$ consisting of $0$s and $1$s with the following properties:

(i) For any $0 \leq \alpha \leq 1$, there is a sequence in $C$ which is $A$-limitable to $\alpha$.

(ii) If $\lim_{i \to \infty} \sum_{j=1}^n a_{ij} x_j = \lim_{i \to \infty} \sum_{j=1}^n a_{ij} x_j = \alpha$ where $(x_i) \in C$, then $\lim_{i \to \infty} \sum_{j=1}^n a_{ij} x_j = \alpha \beta$ if $(x_i) \in C$ and is $A$-limitable to $\beta$.

Then for any $\xi$ in the convex hull of limit points of a bounded sequence $s_n$ in $E^p$, there is a subsequence which is $A$-limitable to $\xi$.

**Proof.** Let $Q$ be the set of limit points of $s_n$, and let $\xi = \sum_{i=1}^n a_i \xi_i$ where $\xi_i \in Q$, $a_i > 0$, and $\sum_{i=1}^n a_i = 1$. Proceed by induction on $m$. If
m = 1, then simply choose a subsequence of $s_n$ which converges to $\xi$. Let $a = \sum_{i=1}^{n-1} a_i$, $b_i = a_i/a$, $\eta = \sum_{i=1}^{n-1} b_i \xi_i$ so that $\xi = \eta a + a_0 \xi_0$. There is a subsequence $s_{n_0}$ of $s_n$ which is A-limitable to $\eta$. Also there is sequence $\epsilon_n$ in $C$ so that

$$\lim_{i \to \infty} a_i \epsilon_i = \lim_{i \to \infty} a_i \epsilon_i = a.$$  

Let

$$y_m = \begin{cases} s_{n_0} & \text{if } m = j_s, \\ \xi & \text{otherwise.} \end{cases}$$

Now let $y'_m$ be the sequence obtained from $y_m$ replacing the terms which equal $\xi_0$ by successive elements of $s_n$ which converges to $\xi_0$ so that $y'_m$ is a subsequence of $s_n$. By induction it is easy to see that $y'_m$ thus constructed is A-limitable to $\xi$.

The next problem is to find a set $C$ of Lemma 1 for $C_l$- and $E_l$-processes.

**Lemma 2.** There is a set of sequences $C$ of Lemma 1 for $C_l$- and $E_l$-processes.

**Proof.** Let $\alpha \in [0, 1]$ and $\alpha = a_0 a_1 a_2 \cdots$ be a binary representation of $\alpha$. For each positive integer $k \geq 2$ define a sequence $s^k$ as follows,

$$s^k_{2n-1} = 0, \quad s^k_{2n} = 0, \quad s^k_{2^k} = 1, \quad s^k_{2^k} = 0$$

and for $k = 1$, define $s^1_{2n-1} = 1, s^1_{2n} = 0$. Then $s^k$ is $C_l$-limitable to $\frac{1}{2^k}$ for each $k$, and $s^k = 1$ for each $n$. Now the sequence defined by $s_n = \sum_{i=1}^{n-1} s^k = 1$ for each $n$. Now the sequence defined by $s_n = \sum_{i=1}^{n-1} a_i s^k$ consists of the elements 0 and 1, and $C_l$-limitable to $\alpha$. The condition (ii) of Lemma 1 can easily be verified. Note that every sequence used in the proof satisfies the condition $(s_1 + \cdots + s_n)/n = \xi + o(1/\sqrt{n})$ where $\xi$ is the $C_l$-limit of $s_n$; hence it is also $E_l$-limitable to $\xi$. Lemma 1 and Lemma 2 give the following theorem.

**Theorem 2.** Let $s_n$ be any bounded sequence in $E^p$. Let $\xi$ be any element in the convex hull of limit points of $s_n$, then there is a subsequence of $s_n$ which is $C_l$- and $E_l$-limitable (hence also Abel and Borel limitable) to $\xi$.

In general it is not possible to extend the result to unbounded sequences. For example $0, 1, 0, 2, 0, 3, \cdots$ has no $C_l$-limitable except the trivial ones. However, it can be shown that the sequence can be rearranged so that the resulting sequence is $C_l$-limitable to
any pre-assigned nonnegative number. But the sequence 0, 1!, 0, 2!, 0, 3!, ... does not even have this property. In general if \( s_n \) is a sequence having a limit point \( \xi \) and it has a subsequence \( k_n \to \infty \) with \( k_n/(k_1+\cdots+k_n)\to 0 \), then for \( \xi < \alpha < \infty \) it is possible to find a rearrangement of \( s_n \) whose \( C_1 \)-limit is \( \alpha \). Without loss of generality assume \( \xi = 0 \). Let \( \lfloor (k_1+\cdots+k_n)/\alpha \rfloor \) be the greatest integer less than or equal to \( (k_1+\cdots+k_n)/\alpha \). Construct a sequence \( y_n \) inserting 0s in \( k_1, k_2, \cdots \) so that the number of 0s preceding \( k_n \) is \( \lfloor (k_1+\cdots+k_n)/\alpha \rfloor \). Then \( y_n \) is \( C_1 \)-limitable to \( \alpha \). Now there is a subsequence \( s_{n_k} \) of \( s_n \) which converges to 0. Replace each element in \( y_n \) which equals 0 by successive elements of \( s_{n_k} \). Then the sequence \( y_n' \) thus constructed is \( C_1 \)-limitable to \( \alpha \). Now insert the rest of the elements of \( s_n \) in \( y_n' \) occasionally so that the resulting sequence has \( C_1 \)-limit \( \alpha \). This remark can be made also in \( E^p \) without much difficulty.

References


Rutgers University and
University of Pittsburgh