Let \( s_n \) be a sequence in a \( p \)-dimensional Euclidean space \( E^p \). Let \( K_n = K(s_n, s_{n+1}, \ldots) \) be the convex hull of \( s_n, s_{n+1}, \ldots \) and \( \overline{K}_n \) its closure. The core of \( s_n \) is defined as \( \bigcap_{n=1}^{\infty} \overline{K}_n \). Knopp’s core theorem states that if \( A = (a_{ij}) \) is an infinite regular matrix with nonnegative elements, then the core of the \( A \)-transform of \( s_n \) is contained in the core of \( s_n \). In particular if \( s_n \) is bounded, every \( A \)-limit of a subsequence of \( s_n \) is contained in the convex hull of limit points of \( s_n \). With certain restrictions on \( A \), the converse is also true; i.e., for any element \( \xi \) in the convex hull of limit points of \( s_n \), there is a subsequence of \( s_n \) which is \( A \)-limitable to \( \xi \). The main objective of this paper is to show that for any \( \xi \) in the convex hull of limit points of a bounded sequence \( s_n \), there is a subsequence of \( s_n \) which is \( C_1 \)- and \( E_1 \)-limitable to \( \xi \).

The following is Knopp’s core theorem in \( E^p \).

**Theorem 1.** Let \( s_n \) be a sequence in \( E^p \) and \( A = (a_{ij}) \) a regular matrix with \( a_{ij} \geq 0 \). Let \( K_n \) be the convex hull of \( s_n, s_{n+1}, \ldots \) and \( K_n' \) the convex hull of \( s_n', s_{n+1}', \ldots \), where \( s_n' = \sum_{j=1}^{\infty} a_{nj}s_j \) is defined for each \( n = 1, 2, \ldots \). Then \( \bigcap_{n=1}^{\infty} K_n' \subset \bigcap_{n=1}^{\infty} \overline{K}_n \).

**Proof.** For each \( \varepsilon > 0 \), define \( K_n = \{ x \mid \inf_{y \in K_n} \| x - y \| \leq \varepsilon \} \). For given \( m \) and \( \varepsilon > 0 \), choose \( n(m, \varepsilon) \) so that
\[
\left| \sum_{j=m+1}^{\infty} a_{nj}s_j \right| < \varepsilon, \quad \left| \sum_{j=m+1}^{\infty} a_{nj} - 1 \right| < \varepsilon, \quad \left| \sum_{j=m}^{\infty} a_{nj} \right| < \varepsilon \quad \text{for } n \geq n(m, \varepsilon).
\]

Now choose an index \( k(n) \) so that
\[
\left| \sum_{j=m+k+1}^{\infty} a_{nj}s_j \right| < \varepsilon, \quad \left| \sum_{j=m+k+1}^{\infty} a_{nj} \right| < \varepsilon.
\]

Then \( \sum_{j=m}^{n+k+1} a_{nj} - 1 \leq 3\varepsilon \). Assume \( \varepsilon < 1/3 \) so that \( \sum_{j=m}^{n+k+1} a_{nj} \neq 0 \). Let \( s_n' = \alpha_n + \beta_n + \gamma_n \) where
\[
\alpha_n = \sum_{j=1}^{m-1} a_{nj}s_j, \quad \beta_n = \sum_{j=m+k+1}^{\infty} a_{nj}s_j, \quad \gamma_n = \sum_{j=m+k+1}^{\infty} a_{nj}.
\]

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and

\[ x_n = \frac{\sum_{j=m}^{m+k} a_n s_j}{\sum_{j=m}^{m+k} a_n} \]

Since \( x_n \in K_m \) for every \( \epsilon > 0 \) and \( s_n' / \gamma_n \in K_{m'} \) for \( n \geq \nu \) where \( \epsilon' = 2\epsilon / (1 - 3\epsilon) \), \( |\gamma_n - 1| < 3\epsilon \).

Now let \( \xi \in K_m' \) for every \( n \). Then \( \xi \in K_{m'} \) for every \( \delta > 0 \) and every \( n \); therefore \( \xi = a_1 s_{n_1} + \cdots + a_k s_{n_k} + \eta \) where \( \|\eta\| < \delta \), \( n_i \geq \nu \) for each \( i \), \( a_i > 0 \) and \( \sum_{i=1}^{k} a_i = 1 \). Since \( s_n' / \gamma_n \in K_{m'} \) for \( n \geq \nu \), \( \xi = \gamma + \sum_{i=1}^{k} a_i x_{n_i} \gamma_{n_i} \) where \( x_{n_i} \in K_{m'} \) for \( n_i \geq \nu \) and \( |\gamma_{n_i} - 1| < 3\epsilon \). Let \( \gamma = \sum_{i=1}^{k} a_i \gamma_{n_i} \), then \((1/\gamma)(\sum_{i=1}^{k} a_i \gamma_{n_i} x_{n_i}) \in K_{m'}\); hence \( \xi \in K_m \) for every \( m \).

It is clear that \( \cap_{n=1}^{\infty} K_n \) always contains limit points of \( s_n \); hence it contains the convex hull of limit points of \( s_n \). Moreover if \( s_n \) is bounded \( \cap_{n=1}^{\infty} K_n \) is precisely the convex hull of limit points of \( s_n \). To see this let \( Q \) be the set of limit points of \( s_n \) and \( N_\delta(x) \) the \( \epsilon \)-neighborhood of \( x \). Then for each \( \epsilon > 0 \), \( \cup_{\delta \in Q} N_\delta(x) \) contains all but a finite number of \( s_n \). Now choose an index \( p \) so that \( s_n \in \bigcup_{\delta \in Q} N_\delta(x) \) for \( n \geq p \). Moreover since \( \xi \in K_{m'} \) for every \( \delta > 0 \) and \( m \), \( \xi = \sum_{i=1}^{k} a_i s_{n_i} + \eta \) where \( a_i \geq 0 \), \( \sum_{i=1}^{k} a_i = 1 \), \( n_i \geq \nu \) and \( \|\eta\| < \delta \). Therefore there are \( \xi_1, \xi_2, \ldots, \xi_k \) in \( Q \) such that \( \xi = \gamma + \sum_{i=1}^{k} a_i \xi_i + \sum_{i=1}^{k} a_i \xi_i \) where \( \|s_k\| \leq \epsilon \); therefore, \( \xi \in K_{m+k}(Q) \). Hence \( \xi \in K(Q) \). The above remark and Knopp’s core theorem give the following corollary.

**Corollary 1.** If \( A \) is an infinite regular matrix with nonnegative elements, then every \( A \)-limit of a bounded sequence is in the convex hull of the limit points of the sequence.

The following is a sufficient condition on a regular matrix so that the converse of Corollary 1 holds.

**Lemma 1.** Let \( A \) be a nonnegative regular matrix. Suppose there is a set of sequences \( \mathbf{C} \) consisting of 0s and 1s with the following properties;

(i) For any \( 0 \leq \alpha \leq 1 \), there is a sequence in \( \mathbf{C} \) which is \( A \)-limitable to \( \alpha \).

(ii) If \( \lim_{i \to \infty} \sum_{j=1}^{n} a_{ij} x_j = \lim_{i \to \infty} \sum_{j=1}^{n} a_{ij} = \alpha \) where \( (x_i) \in \mathbf{C} \), then \( \lim_{i \to \infty} \sum_{j=1}^{n} a_{ij} x_j = a \beta \) if \( (x_i) \in \mathbf{C} \) and is \( A \)-limitable to \( \beta \).

Then for any \( \xi \) in the convex hull of limit points of a bounded sequence \( s_n \) in \( E^p \), there is a subsequence which is \( A \)-limitable to \( \xi \).

**Proof.** Let \( Q \) be the set of limit points of \( s_n \), and let \( \xi = \sum_{i=1}^{n} a_i \xi_i \) where \( \xi_i \in Q \), \( a_i > 0 \), and \( \sum_{i=1}^{n} a_i = 1 \). Proceed by induction on \( m \). If
Let $a = \sum_{i=1}^{\infty} a_i$, $b_i = a_i/a$, $\eta = \sum_{i=1}^{\infty} b_i \xi_i$ so that $\xi = \eta a + a_0 \xi_0$. There is a subsequence $s_{n_r}$ of $s_n$ which is $A$-limitable to $\eta$. Also there is sequence $e_n$ in $C$ so that

$$\lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} e_j = \lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} = a.$$

Let

$$y_m = \begin{cases} s_{n_r} & \text{if } m = j_r, \\ \xi_k & \text{otherwise.} \end{cases}$$

Now let $y'_m$ be the sequence obtained from $y_m$ replacing the terms which equal $\xi_k$ by successive elements of $s_n$ which converges to $\xi_k$ so that $y'_m$ is a subsequence of $s_n$. By induction it is easy to see that $y'_m$ thus constructed is $A$-limitable to $\xi$.

The next problem is to find a set $C$ of Lemma 1 for $C_1$- and $E_1$-processes.

**Lemma 2.** There is a set of sequences $C$ of Lemma 1 for $C_1$- and $E_1$-processes.

**Proof.** Let $a \in [0, 1]$ and $a = 0 \cdot a_1 a_2 \cdots$ be a binary representation of $a$. For each positive integer $k \geq 2$ define a sequence $s_k$ as follows,

$$s_k(2n-1) = 0, \quad s_k(2n) = 0, \ldots, s_k(2k-1) = 1, \quad s_k(2k) = 0$$

and for $k = 1$, define $s_1_{2n-1} = 1$, $s_1_{2n} = 0$. Then $s_k$ is $C_1$-limitable to $1/2^k$ for each $k$, and $\sum_{n=1}^{\infty} s_n = 1$ for each $n$. Now the sequence defined by $s_n = \sum_{k=1}^{\infty} s_n s_k$ consists of the elements $0$ and $1$, and $C_1$-limitable to $a$. The condition (ii) of Lemma 1 can easily be verified. Note that every sequence used in the proof satisfies the condition $(s_1 + \cdots + s_n)/n = \xi + o(1/\sqrt{n})$ where $\xi$ is the $C_1$-limit of $s_n$; hence it is also $E_1$-limitable to $\xi$. Lemma 1 and Lemma 2 give the following theorem.

**Theorem 2.** Let $s_n$ be any bounded sequence in $E^n$. Let $\xi$ be any element in the convex hull of limit points of $s_n$, then there is a subsequence of $s_n$ which is $C_1$- and $E_1$-limitable (hence also Abel and Borel limitable) to $\xi$.

In general it is not possible to extend the result to unbounded sequences. For example $0, 1, 0, 2, 0, 3, \cdots$ has no $C_1$-limitable except the trivial ones. However, it can be shown that the sequence can be rearranged so that the resulting sequence is $C_1$-limitable to
any pre-assigned nonnegative number. But the sequence 0, 1!, 0, 2!, 0, 3!, ... does not even have this property. In general if \( s_n \) is a sequence having a limit point \( \xi \) and it has a subsequence \( k_n \to \infty \) with \( k_n/(k_1 + \cdots + k_n) \to 0 \), then for \( \xi < \alpha < \infty \) it is possible to find a rearrangement of \( s_n \) whose \( C_1 \)-limit is \( \alpha \). Without loss of generality assume \( \xi = 0 \). Let \( \lfloor (k_1 + \cdots + k_n)/\alpha \rfloor \) be the greatest integer less than or equal to \( (k_1 + \cdots + k_n)/\alpha \). Construct a sequence \( y_n \) inserting 0s in \( k_1, k_2, \ldots \) so that the number of 0s preceding \( k_n \) is \( \lfloor (k_1 + \cdots + k_n)/\alpha \rfloor \). Then \( y_n \) is \( C_1 \)-limitable to \( \alpha \). Now there is a subsequence \( s_{n_k} \) of \( s_n \) which converges to 0. Replace each element in \( y_n \) which equals 0 by successive elements of \( s_{n_k} \). Then the sequence \( y'_n \) thus constructed is \( C_1 \)-limitable to \( \alpha \). Now insert the rest of the elements of \( s_n \) in \( y'_n \) occasionally so that the resulting sequence has \( C_1 \)-limit \( \alpha \). This remark can be made also in \( E^p \) without much difficulty.

References


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