MULTIPLIER TRANSFORMATIONS. III

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Let $N_a, -1/2 < \alpha < 1/2$, be the space of those complex valued functions $F(n), n \in I$ the additive group of integers, for which $N_a[F]$ is finite where

$$N_a[F] = \left[ \sum_{-\infty}^{\infty} |F(n)|^2 |n + 1|^{2\alpha} \right]^{1/2}.$$

If $F \in N_0$ the Fourier transform $(M_2)$

$$\hat{F}^\ast(\theta) = \sum_{-\infty}^{\infty} F(n) e^{2\pi i n \theta},$$

is defined as a limit in the mean of order 2 for $\theta \in T$, where $T$ is the additive group of real numbers modulo 1. For such $F$ the following inversion formula is valid,

$$F(n) = \int_T \hat{F}^\ast(\theta) e^{-2\pi i n \theta} d\theta.$$

Let $t(\theta)$ be a bounded measurable function on $T$ and let us define

$$TF^\ast(n) = \int_T \hat{F}^\ast(\theta) t(\theta) e^{-2\pi i n \theta} d\theta$$

for $n \in I$ and $F \in N_0$. If

$$N_a[T] = 1.0.1.b. \{ N_a[TF] / N_a[F], F \in N_a \cap N_0, F \neq 0 \}$$

is finite then, since $N_a \cap N_0$ is dense in $N_a$, $T$ has a unique extension as a bounded linear transformation (with norm $N_a[T]$) of $N_a$ into itself. The problem with which we are concerned is that of finding sufficient conditions on the multiplier function $t(\theta)$ which will insure that the corresponding multiplier transformation $T$ is bounded on $N_a$. In the present paper, which continues investigations begun in $[1; 2; 3]$, we will obtain a sufficient condition involving $\beta$-variation.

A function $f(x)$ defined on $I = \{ a \leq x \leq b \}$ is said to be of bounded $\beta$-variation (1 $\leq \beta < \infty$) if $V_\beta[f, I] = V_\beta[f]$ is finite where

$$V_\beta[f] = 1.0.1.b. \left[ \sum_{k=0}^{n} |f(x_{k+1}) - f(x_k)|^\beta \right]^{1/\beta}.$$

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Here the least upper bound is taken over all finite sets \( a \leq x_0 < x_1 < \cdots < x_n \leq b \). Note that if \( \beta_2 > \beta_1 \)

\[
V_{\beta_1}[f]^{\beta_1} \leq (2||f||_{\infty})^{\beta_2 - \beta_1} V_{\beta_1}[f]^{\beta_1}.
\]

Thus if \( V_{\beta_1}[f] \) is finite then so is \( V_{\beta_2}[f] \). Our principal result is that if \( V_{\beta_1}[t(\theta)] \) is finite, \( \beta \geq 2 \), then \( N_\alpha[T] \) is finite for \( |\alpha| < 1/\beta \). This result is of interest in that the entire permissible range of \( \alpha, -1/2 < \alpha < 1/2 \), is already obtained for \( \beta = 2 \).

We begin by recalling several results from [2] and [3]. Let \( 0 < \alpha < 1/2 \) be fixed. Note that for \( \alpha > 0 \), \( N_\alpha \subset N_\gamma \) so that \( F^*(\theta) \) is well defined for every \( F \in N_\alpha \). For \( t(\theta) \) a bounded measurable function on \( T \) we define \( A_\alpha[t] \) to be the smallest constant such that

\[
\int_T \int_T |F^*(\theta)|^2 |t(\theta) - t(\phi)|^2 \sin \pi(\theta - \phi)|^{-1-2\alpha} d\theta d\phi \leq A_\alpha[t] N_\alpha[F]^2
\]

for every \( F \in N_\alpha \). \( A_\alpha[t] \) may of course be \( +\infty \).

**Lemma 1.** For \( 0 < \alpha < 1/2 \) we have

\[
N_\alpha[T] \leq 2 \max\{|d|, A_\alpha[t]\}.
\]

**Lemma 2.** For \( 0 < \alpha < 1/2 \) there is a constant \( A(\alpha) \) depending only on \( \alpha \) such that for any \( \psi \) in \( T \)

\[
\int_T \left| F(\theta) \right|^2 \sin \pi(\theta - \phi)|^{-2\alpha} d\theta \leq A(\alpha) N_\alpha[F]^2.
\]

**Lemma 3.** Let \( f(x) \) be a real function defined on the interval \( I = \{ a \leq x \leq b \} \). For each \( \beta > 1 \) there exists a constant \( C(\beta) \) depending only on \( \beta \) such that for each \( f \) for which \( V_\beta[f] < \infty \) and each \( \epsilon > 0 \) there exists \( f_\epsilon(x) \) with the properties:

a. \( ||f - f_\epsilon||_\infty \leq \epsilon \quad x \in I \);

b. \( V_1[f_\epsilon] \leq C(\beta) V_\beta[f]^{1-\beta} \).

Here \( ||\cdot||_\infty \) is the uniform norm on \( I \). This is proved in [2] under the added assumption that \( f(x) \) is continuous. However a simple modification of the proof given there shows that this assumption is unnecessary.

**Lemma 4.** Suppose that \( 0 < \alpha < 1/2 \). Let \( t(\theta) \) be of bounded 1-variation on \( T \). Then if \( T \) is the corresponding multiplier transformation we have

\[
N_\alpha[T]^2 \leq B(\alpha) \{||d||_\infty^2 + ||d||_\infty V_1[t]\},
\]

where \( B(\alpha) \) is a finite constant depending only on \( \alpha \).

**Proof.** We begin by proving this result under the assumption that
\( \phi(\theta) \) is in addition continuous. At the end this restriction will be removed, using a standard approximation argument. For \( F \in \mathcal{N}_a \) consider the quantity

\[
Q = \int_T F^-(\theta)^2 d\theta \int_T |\phi(\theta) - \phi(\phi)| \sin \pi(\theta - \phi)^{-1-2a} d\phi
\]

\[
\leq 2||d||_\infty \int_T |F^-(\theta)|^2 d\theta \int_T |\phi(\theta) - \phi(\phi)| \sin \pi(\theta - \phi)^{-1-2a} d\phi.
\]

We have

\[
\int_T |\phi(\theta) - \phi(\phi)| \sin \pi(\theta - \phi)^{-1-2a} d\phi \leq I_1 + I_2
\]

where

\[
I_1 = \int_\phi^\phi |\sin \pi(\theta - \phi)^{-1-2a} d\phi \int_\phi^\phi |d\phi|.
\]

\[
I_2 = \int_\phi^\phi |\sin \pi(\theta - \phi)^{-1-2a} d\phi \int_\phi^\phi |d\phi|.
\]

By Fubini's theorem

\[
I_1 = \int_\phi^\phi |d\phi| \int_\phi^\phi |\sin \pi(\theta - \phi)^{-1-2a} d\phi|.
\]

An easy computation shows that there exists a constant \( A_1(\alpha) \) such that if \( \theta \leq \phi \leq \theta + 1/2 \)

\[
\int_\phi^\phi |\sin \pi(\theta - \phi)^{-1-2a} d\phi| \leq A_1(\alpha) |\sin \pi(\theta - \phi)^{-2a}.
\]

Thus

\[
I_1 \leq A_1(\alpha) \int_\phi^\phi |\sin \pi(\theta - \phi)^{-2a} d\phi|,
\]

and similarly

\[
I_2 \leq A_1(\alpha) \int_\phi^\phi |\sin \pi(\theta - \phi)^{-2a} d\phi|.
\]

Making use of these inequalities and using Fubini's theorem we find that

\[
Q \leq 2A_1(\alpha)||d||_\infty \int_T |F^-(\theta)|^2 d\theta \int_T |\sin \pi(\theta - \phi)^{-2a} d\phi|.
\]

\[
\leq 2||d||_\infty A_1(\alpha) \int_T |d\phi| \int_T |F^-(\theta)|^2 \sin \pi(\theta - \phi)^{-2a} d\phi.
\]
Applying Lemma 2 we obtain
\[ Q \leq 2A_1(\alpha)A(\alpha)\|t\|_\infty V_1[t]N_a[F]^2. \]

Thus \( A_a[t] \leq 2A_1(\alpha)A(\alpha)\|t\|_\infty V_1[t] \). Our proof is now complete if \( t(\theta) \) is continuous. If \( t(\theta) \) is not continuous we set
\[
t_n(\theta) = \int_T k_n(\theta - \phi)t(\phi)d\phi \quad n = 1, 2, \ldots \]

where \( k_n(\theta) \) is any sequence of functions on \( T \) satisfying:

i. \( k_n(\theta) \) is continuous;

ii. \( k_n(\theta) \geq 0, \int_T k_n(\theta)d\theta = 1; \)

iii. \( \lim_{n \to \infty} \int_U k_n(\theta)d\theta = 1 \), for any fixed open set \( U \) in \( T \) which contains 0.

With these assumptions it is easily verified that:

i. \( \|t_n\|_\infty \leq \|t\|_\infty; \)

ii. \( V_1[t_n] \leq V_1[t]; \)

iii. \( \lim_{n \to \infty} t_n(\theta) = t(\theta) \) for all \( \theta \) at which \( t(\cdot) \) is continuous.

Let \( T_n \) be the multiplier transform generated by \( t_n(\theta) \); then
\[
N_a[T] \leq \liminf_{n \to \infty} N_a[T_n].
\]

Since \( t_n(\theta) \) is continuous
\[
N_a[T_n]^2 \leq B(\alpha)\{\|t_n\|_\infty^2 + \|t_n\|_\infty V_1[t_n]\},
\]
\[
\leq B(\alpha)\{\|t\|_\infty^2 + \|t\|_\infty V_1[t]\}.
\]

Combining these results our desired lemma follows.

**Theorem.** Let \( t(\theta) \) be defined for \( \theta \in T \) and let \( T \) be the corresponding multiplier transformation. If \( V_\beta[t] \) is finite (where \( \beta > 2 \)) then
\[
N_a[T] < \infty \quad \text{if} \quad \alpha < 1/\beta.
\]

**Proof.** By Lemma 3 there exists a sequence of functions \( s_n(\theta) \), \( \theta \in T \), such that
\[
\|s_n - t\|_\infty \leq 2^{-n},
\]
\[
V_1[s_n] \leq C(\beta)V_\beta[f]2^{\beta(\beta-1)} \quad n = 1, 2, \ldots.
\]

Let
\[
t_1(\theta) = s_1(\theta),
\]
\[
t_n(\theta) = s_n(\theta) - s_{n-1}(\theta) \quad n = 2, 3, \ldots.
\]
Then

$$l(\theta) = \sum_{n}^{\infty} l_n(\theta),$$

and thus by an evident argument

$$N_n[T] \leq \sum_{n}^{\infty} N_n[T_n]$$

where $T_n$ is the multiplier transformation generated by the multiplier function $l_n(\theta)$. We have

$$\|l_n\|_\infty = O(2^{-n}) \quad n = 1, 2, \ldots,$$

$$\mathcal{V}_1[T_n] = O(2^{n(\beta-1)}) \quad n = 1, 2, \ldots.$$  

Choose $\gamma, \alpha < \gamma < 1/2$. By Lemma 4

$$N_\gamma[T_n] = O\left[(2^{-n})^2 + 2^{-n}2^{n(\beta-1)}\right]^{1/2},$$

$$= O(2^{n(\beta/2-1)}).$$

On the other hand by Parseval’s equality

$$N_\delta[T_n] = \|l_n\|_\infty = O(2^{-n}).$$

Applying the Riesz-Thorin convexity theorem we find that if $\alpha = (1-\theta)0 + \theta\gamma$ then

$$N_\alpha[T_n] = O(2^{-n(1-\theta)2^{n(\beta/2-1)\gamma}}),$$

$$= O(2^{n(-1+\beta\gamma/2\gamma)}).$$

Thus the series $\sum_n^{\infty} N_\alpha[T_n]$ is convergent if $\beta\alpha/2\gamma < 1$; that is if $\alpha < 2\gamma/\beta$. Since $\gamma$ is arbitrary subject to the restriction $\alpha < \gamma < 1/2$, it is always possible to choose $\gamma$ so that $\alpha < 2\gamma/\beta$ if $0 < \alpha < 1/\beta$. Thus our theorem is true if $0 < \alpha < 1/\beta$. The case $-1/\beta < \alpha < 0$ follows by a familiar duality argument, while the case $\alpha = 0$ is trivial.

For $f(x)$ defined on the interval $I$ let $W_\beta[f, I]$ be the smallest constant such that for every $\varepsilon > 0$ there exists a function $f_\epsilon(x), x \in I$, satisfying:

a. $\|f - f_\epsilon\|_\infty \leq \varepsilon,

b. \mathcal{V}_1[f_\epsilon, I] \leq W_\beta[f, I] \varepsilon^{1-\beta}.$

$W_\beta[f, I]$ can of course be $\infty$. Lemma 3 asserts that

$$W_\beta[f, I] \leq C(\beta) \mathcal{V}_\beta[f, I] \varepsilon^{\delta}.$$  

(1)

The assumption in our principal theorem that $\mathcal{V}_\beta[f, T] < \infty$ is made only to insure that $W_\beta[f, T] < \infty$. The following lemma shows that
the assumption $W_\beta[f, T] < \infty$ is "almost" as strong as the assumption $V_\beta[f, T] < \infty$.

**Lemma 5.** For each $\beta$, $1 \leq \beta < \infty$, and each $\gamma > \beta$ there exists a finite constant $A(\beta, \gamma)$ such that

\[ V_\gamma[f, I] \leq A(\beta, \gamma) \|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f, I]^{1/\gamma}. \]

**Proof.** For each $k = 0, 1, \ldots$ let $f_k$ satisfy

\[ \|f - f_k\|_\infty \leq 2^{-k} \|f\|_\infty, \]

\[ V_1[f_k] \leq W_\beta[f] 2^{k(\beta-1)} \|f\|_\infty^{1-\beta}. \]

If we define

\[ g_0(x) = f_0(x), \]

\[ g_k(x) = f_k(x) - f_{k-1}(x) \quad k = 1, 2, \ldots, \]

then

\[ \sum_{k=0}^{\infty} g_k(x) = f(x) \quad x \in I. \]

Moreover

\[ \|g_k(x)\|_\infty \leq 4 2^{-k} \|f\|_\infty, \]

\[ V_1[g_k] \leq 2 W_\beta[f] 2^{k(\beta-1)} \|f\|_\infty^{1-\beta}. \]

It is easy to see using Hölder’s inequality that $V_\gamma[f] \leq \sum_0^\infty V_\gamma[g_k]$. Also

\[ V_\gamma[g]^\gamma \leq (2 \|g\|_\infty)^{\gamma-1} V_1[g]. \]

Thus

\[ V_\gamma[f] \leq \sum_{0}^{\infty} (2^{-k} \|f\|_\infty)^{(\gamma-1)/\gamma} (2 W_\beta[f] 2^{k(\beta-1)} \|f\|_\infty^{1-\beta})^{1/\gamma} \]

\[ \leq \|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f]^{1/\gamma} 2^{(3\gamma-2)/\gamma} \sum_{0}^{\infty} 2^{-k(\gamma-\beta)/\gamma} \]

\[ \leq A(\beta, \gamma) \|f\|_\infty^{(\gamma-\beta)/\gamma} W_\beta[f]^{1/\gamma}. \]

On the other hand the assumption $W_\beta[f] < \infty$ is slightly weaker than the assumption $V_\beta[f] < \infty$ in that for $\beta > 1$ no inequality of the form

\[ V_\beta[f, I]^{\beta} \leq A(\beta) W_\beta[f, I] \]
is true for all \( f \). To see this let us set \( I_k = \{ k - 1 \leq x \leq k \} \) for all \( k = 0, 1, 2, \ldots \), and \( I_N = I_1 \cup I_2 \cup \cdots \cup I_N \). Further let

\[
f_1(x) = 2^{-k/\beta} \sin[2^k(2\pi x)]
\]

\( x \in I_k \).

Very simple computations show that there are positive constants \( c_1(\beta) \) and \( c_2(\beta) \) independent of \( k \) and \( N \) such that

\[
W_\beta[f_1, I_k] \geq c_1(\beta) \qquad k = 1, 2, \ldots ,
\]

\[
W_\beta[f_1, I_N] \leq c_2(\beta) \qquad N = 1, 2, \ldots .
\]

It is evident that

\[
\sum_{1}^{N} V_\beta[f, I_k] \leq V_\beta[f, I_N].
\]

If (3) held then using (1) we would have

\[
C(\beta)^{-1} \sum_{1}^{N} W_\beta[f, I_k] \leq A'(\beta)W_\beta[f, I_N].
\]

However for \( f = f_1 \) and for \( N \) sufficiently large this is impossible.

REFERENCES