ABSOLUTE CONTINUITY OF CERTAIN UNITARY AND HALF-SCATTERING OPERATORS

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1. The theorem. Let $A$ be a self-adjoint operator, bounded from below, and $D$ a bounded, non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$ for which the set

\[ \mathcal{D}_A \cap \mathbb{R}^{1/d} \text{ is dense.} \]

Suppose that $A + D$ is unitarily equivalent to $A$ and that $U$ is any unitary operator effecting this equivalence, thus

\[ A + D = UAU^*. \]

Then $U$ is absolutely continuous, that is, if $U$ has the spectral resolution

\[ U = \int_0^{2\pi} e^{ik}dE(\lambda), \]

and if $x$ is an arbitrary element of the Hilbert space, then $\|E(\lambda)x\|^2$ is an absolutely continuous function of $\lambda$.

If $A$ is bounded, then $\mathcal{D}_A = \mathcal{H}$ and (1) reduces to the assumption that $0$ cannot be in the point spectrum of $D$. In this case, the assertion of the theorem was proved in [2]. In [3], there were obtained lower estimates for the measure of the spectrum of $U$ both when $A$ was bounded and also in the case when $A$ was supposed only half-bounded. It will be shown in the present paper that the methods used in this latter case will also yield the absolute continuity of $U$ under the conditions specified in the theorem.

An application to half-scattering operators will be given in §3.

2. Proof of the theorem. Since (1) and (2) hold if $A$ is replaced by $A + cI$, where $c = \text{const.}$, it is clear that there is no loss of generality in assuming that

\[ A \geq 0. \]

If $f(\lambda)$ is a real-valued function of period $2\pi$ with a continuous first derivative and having the Fourier series

\[ f(\lambda) = \sum_{k=-\infty}^{\infty} c_ke^{i\lambda}, \quad c_k = (2\pi)^{-1} \int_0^{2\pi} f(\lambda)e^{-ik}\lambda}d\lambda, \quad c_{-k} = c_k, \]

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then it follows from (3) and (5) that

$$D^{1/2} \int_0^{2\pi} f(\lambda) dE(\lambda) = c_0 D^{1/2} + \sum_{k=1}^{\infty} c_k D^{1/2} U^k + \sum_{k=1}^{\infty} c_k D^{1/2} U^{*k}. \tag{6}$$

Next, let $y$ be in $\mathcal{D}_A \cap \mathcal{M} D^{1/2}$, so that $y = D^{1/2}x$ is in $\mathcal{D}_A$. It follows from (6) that

$$\left( x, D^{1/2} \int_0^{2\pi} f(\lambda) dE(\lambda) y \right) = (x, c_0 D^{1/2} y) + 2 \text{Re} \left( x, \sum_{k=1}^{\infty} c_k D^{1/2} U^k y \right). \tag{7}$$

Since $(\text{Re}(\cdots))^2 \leq \|x\|^2 (\sum_{k=1}^{\infty} |c_k|^2)(\sum_{k=1}^{\infty} \|D^{1/2} U^k y\|^2)$, an application of the Schwarz inequality to (7) yields

$$\left( \int_0^{2\pi} f(\lambda) d\|E(\lambda) y\|^2 \right)^2 \leq 2 \left[ |c_0|^2 \|y\|^4 + 4 \|x\|^2 \left( \sum_{k=1}^{\infty} |c_k|^2 \right) \left( \sum_{k=1}^{\infty} \|D^{1/2} U^k y\|^2 \right) \right]. \tag{8}$$

But (2) implies that for $n = 1, 2, \cdots, \sum_{k=1}^{n} U^{*k} U^k = A - U^{*n} A U^n \leq A$, the inequality by (4), and so $\sum_{k=1}^{n} \|D^{1/2} U^k y\|^2 \leq (A, y, y)$. Relation (8) and the Parseval relation $(2\pi)^{-1} \int_0^{2\pi} f^2(\lambda) d\lambda = |c_0|^2 + 2 \sum_{k=1}^{\infty} |c_k|^2$ now imply

$$\left( \int_0^{2\pi} f(\lambda) d\|E(\lambda) y\|^2 \right)^2 \leq C(x) \int_0^{2\pi} f^2(\lambda) d\lambda, \tag{9}$$

where $C(x)$ is a number which depends on $x$ (and $y = D^{1/2}x$) but not on the choice of $f(\lambda)$. Since (9) holds for every smooth function on $[0, 2\pi]$ satisfying $f(0) = f(2\pi)$, and since ordinary Lebesgue measure is absolutely continuous, it follows by a standard argument (involving approximations of characteristic functions of intervals by smooth functions $f(\lambda)$ satisfying $f(0) = f(2\pi)$) that $\|E(\lambda) y\|^2$ is also absolutely continuous. It then follows from (1) that $\|E(\lambda) x\|^2$ is absolutely continuous for all $x$ in the Hilbert space and the proof of the theorem is complete.

3. Half-scattering operators. Let $A$ denote the quantum mechanical (half-bounded) energy operator $-d^2/dx^2$ on the Hilbert space

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$L^2(-\infty, \infty)$ and $D$ be a perturbation potential $V(x)$, where $V(x)$ is continuous and satisfies

$$0 \leq V(x) < \text{const.}, \quad -\infty < x < \infty,$$

and

$$\int_{-\infty}^{\infty} V(x)dx < \infty.$$

Then $A$ and $A+D$ are absolutely continuous and each has the half-line $0 \leq \lambda < \infty$ as spectrum (cf. [4] and the references there to Weyl, Kodaira, Titchmarsh). Moreover, it follows from results of Kuroda [1] (cf. [4]) that the half-scattering operators

$$U_+ = \lim_{t \to \infty} U_t \quad \text{and} \quad U_- = \lim_{t \to -\infty} U_t,$$

exist as strong limits satisfying (2). As a consequence of the theorem of the present paper, it follows that if, in addition to (10) and (11), $V(x)$ satisfies

$$V(x) > 0 \text{ almost everywhere on } (-\infty, \infty),$$

then the half-scattering operators $U_+$ and $U_-$ of (12) are absolutely continuous. In fact, (10) implies that $D = V(x)$ is bounded and non-negative while (13) implies (1).

It can be noted that if, in addition to (10), (11) and (13), also

$$\lim_{b \to \infty} \frac{1}{(b-a)^2} \int_{a}^{b} V^{-1}(x)dx = 0,$$

is assumed, then, as was shown in [4], the (absolutely continuous) spectrum of each of the operators $U_+$ and $U_-$ must be the entire unit circle $|z| = 1$.

**References**


