1. Introduction. Let $X$ be a Banach space, $\mathfrak{B}(X)$ the closure of the algebra of all bounded linear operators on $X$ of finite rank, the closure being taken in the topology of the norm

$$
\|B\| = \sup_{x \in X} \frac{\|Bx\|}{\|x\|}, \quad B \in \mathfrak{B}(X).
$$

The present paper is concerned with a generalization of a theorem of F. Bonsall and A. W. Goldie [1] which states that if $X$ is reflexive then $\mathfrak{B}(X)$ is an annihilator algebra.

It is shown that if $X$ is quasi-reflexive the algebra $\mathfrak{B}(X)$ can be written as the direct sum of four closed subalgebras

$$
\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3,
$$

where $\mathfrak{A}_1$ is a right annihilator algebra which is annihilated on the right by the right ideal $\mathfrak{B}_2$ and $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra which is annihilated on the left by the nilpotent algebra $\mathfrak{B}_3$. Moreover if $X$ is reflexive we have $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = (0)$, so that the above mentioned theorem is obtained as a special case of the present result.

2. Definitions and notation. Let $X$ be a Banach space, $X^*$ and $X^{**}$ its first and second conjugate spaces. The symbol $\pi$ will be used to denote the canonical isomorphism of $X$ into $X^{**}$. The annihilator in $X^*$ of a subspace $Y$ of $X$ will be denoted by $Y^+$. If $x \in X$ and $x^* \in X^*$ then as in [4] we use the symbol $x \otimes x^*$ to denote the one dimensional operator on $X$ defined by the equation

$$
(x \otimes x^*)y = x^*(y)x \quad \text{for each } y \in X.
$$

Throughout this paper the closure of the algebra of bounded linear operators of finite rank will be denoted by $\mathfrak{B}(X)$. All algebras of operators under consideration are considered to be normed with the operator bound, that is

$$
\|A\| = \sup_{x \in X} \frac{\|Ax\|}{\|x\|}.
$$

If $A$ is an operator on a Banach space $X$, we denote by $A^*$ the...
adjoint of $A$, that is the operator on $X^*$ defined by $A^*x^*(x) = x^*(Ax)$. The symbol $I$ will denote the identity operator.

Let $\mathfrak{A}$ be an algebra and $\mathfrak{C} \subseteq \mathfrak{A}$. The right (left) annihilator of $\mathfrak{C}$ will be denoted by $R(\mathfrak{C})(L(\mathfrak{C}))$.

An algebra $\mathfrak{A}$ is called a right (left) annihilator algebra [4] if for every closed left (right) ideal $\mathfrak{I}$ we have $R(\mathfrak{I}) = (0)$, $(L(\mathfrak{I}) = (0))$ if and only if $\mathfrak{I} = \mathfrak{A}$.

A Banach space $X$ is called quasi-reflexive of order $n$ [2] if the quotient space $X^{**}/\pi X$ is (finite) $n$-dimensional.

3. Preliminary lemmas.

3.1. Lemma. Let $X$ be a Banach space, $Y_i$, $i = 1, 2$, closed linear subspaces of $X$ such that $X = Y_1 \oplus Y_2$. Let $\mathfrak{B}_i$ be the subset of $\mathfrak{B}(X)$ consisting of those operators in $\mathfrak{B}(X)$ with range contained in $Y_i$, $i = 1, 2$. Then $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$ and $\mathfrak{B}(X) = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

Proof. Let $A \in \mathfrak{B}_i$, $B \in \mathfrak{B}(X)$, then Range $AB \subseteq Y_i$, therefore $\mathfrak{B}_i$ is a right ideal. If $A_n \in \mathfrak{B}_i$ and $A_n \to A$ in the norm topology of $\mathfrak{B}(X)$, then for each $x \in X$, $A_n x \in Y_i$ and $A_n x \to A x$ in the norm topology of $X$. Since $Y_i$ is closed, $A x \in Y_i$, so $A \in \mathfrak{B}_i$. Thus, it follows that $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$. Now, since $X = Y_1 \oplus Y_2$ there exists a continuous projection $P$ of $X$ onto $Y_1$ with null space $Y_2$. If $B \in \mathfrak{B}(X)$ we can write $B = PB + (I - P)B$, where $PB \in \mathfrak{B}_1$ and $(I - P)B \in \mathfrak{B}_2$. This decomposition is obviously unique and therefore $\mathfrak{B}(X) = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

Lemma 2. Let $X$, $Y_1$, $\mathfrak{B}_i$ be as in Lemma 1, let $Z_i$, $i = 1, 2$ be closed linear subspaces of $X^*$ such that $X^* = Z_1 \oplus Z_2$. Let $\mathfrak{A}_i$ be the subset of $\mathfrak{B}_i$ whose elements are the operators in $\mathfrak{B}_i$ whose adjoints have range contained in $Z_i$, $i = 1, 2$. Then $\mathfrak{A}_i$ is a closed left ideal in $\mathfrak{B}_i$ and $\mathfrak{B}_i = \mathfrak{A}_1 \oplus \mathfrak{A}_2$.

Proof. The proof that $\mathfrak{A}_i$ is a closed left ideal is similar to the argument in Lemma 1 and hence is omitted. Let $P$ be the continuous projection of $X$ onto $Y_1$ with null space $Y_2$ and $Q$ the continuous projection of $X^*$ onto $Z_1$ with null space $Z_2$. Note first that $\mathfrak{A}_i$, $i = 1, 2$, are closed subspaces of $\mathfrak{B}_i$ and that if $A_i \in \mathfrak{A}_i$, $i = 1, 2$, we have $\|A_i\| = \|Q(A_1 + A_2)\| \leq \|Q\| \|A_1 + A_2\|$. It follows from Theorem 2.1 [3] that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is a closed subalgebra of $\mathfrak{B}_1$. Next, we show that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is dense in $\mathfrak{B}_1$. A one dimensional operator in $\mathfrak{B}_i$ is of the form $y \otimes x^*$ with $y \in Y$, $x^* \in X^*$ and can be written $y \otimes x^* = y \otimes Qx^* + y \otimes (I - Q)x^*$, with $y \otimes Qx^* \in \mathfrak{A}_1$ and $y \otimes (I - Q)x^* \in \mathfrak{A}_2$. It follows that every operator of finite rank in $\mathfrak{B}_i$ can be written as the sum of an operator in $\mathfrak{A}_1$ and an operator in $\mathfrak{A}_2$. Now, if $B \in \mathfrak{B}_1 \subset \mathfrak{B}(X)$, there exists a sequence $F_n$ of operators of finite rank on $X$ such that $F_n \to B$ in
the norm topology. Since $P$ is a continuous projection on $Y_1$ and Range $B \subset Y_1$ we also have $PF_n \to B$ where $PF_n$ is an operator of finite rank belonging to $\mathcal{B}_1$ and therefore $PF_n \subset \mathcal{A}_1 \oplus \mathcal{A}_2$. It follows that $\mathcal{A}_1 \oplus \mathcal{A}_2$ is dense in $\mathcal{B}_1$ and therefore $\mathcal{A}_1 \oplus \mathcal{A}_2 = \mathcal{B}_1$.

**Lemma 3.** If $X$ is quasi-reflexive of order $n$ then there exists Banach spaces $X_1$ and $X_2$ and an equivalent norm for $X$ such that $X_1^* = X_1$, $X_2^* = X$ and

\begin{align*}
X^* &= \pi_1X_1 \oplus (\pi_2X_2)^+, \\
X &= \pi_2X_2 \oplus V^+, \\
X_1 &= X_2^* = V \oplus U,
\end{align*}

where $\pi_i$ is the canonical embedding of $X_i$ in $X_i^{**}$, $i=1, 2$, and where $U$, $V^+$ and $(\pi_2X_2)^+$ are all (finite) $n$-dimensional.

**Proof.** Since $X$ is quasi-reflexive of order $n$, it follows from Theorem 3.5 of [2] that $X_1$, $X_2$ exist such that $X_1^* = X_1$ and $X_2^* = X$ and such that both $X_1$ and $X_2$ are quasi-reflexive of order $n$. The direct sum decompositions follow at once from Theorem 3.3 and the proof of Theorem 3.1 of [2].

4. **Theorem.** Let $X$ be a quasi-reflexive space. Then $\mathcal{B}(X) = \mathcal{A}_1 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_3$ where $\mathcal{A}_1$ is a right annihilator algebra and $\mathcal{B}_2$ a right ideal, which annihilates $\mathcal{A}_1$ on the right; $\mathcal{A}_1 \oplus \mathcal{B}_1$ is a left annihilator algebra and $\mathcal{B}_3$ a nilpotent algebra which annihilates $\mathcal{A}_1 \oplus \mathcal{B}_1$ on the left. Moreover the following are equivalent:

(a) $\mathcal{A}_1$ is a left annihilator algebra,
(b) $\mathcal{B}_i = (0)$, $i=1, 2, 3$,
(c) $X$ is reflexive.

**Proof.** By Lemma 3 there exist Banach spaces $X_1$ and $X_2$ such that

\begin{align*}
X^* &= \pi_1V \oplus \pi_1U \oplus (\pi_2X_2)^+ \quad \text{and} \quad X = \pi_2X_2 \oplus V^+.
\end{align*}

Let $\mathcal{A}_1$ denote the subalgebra of $\mathcal{B}(X)$ whose elements are the operators in $\mathcal{B}(X)$ with range in $\pi_2X_2$ and which have adjoints with range in $\pi_1V$; let $\mathcal{B}_1$ denote the subalgebra of $\mathcal{B}(X)$ whose elements are those operators in $\mathcal{B}(X)$ with range in $\pi_2X_2$, which have adjoints with range in $\pi_1U$; let $\mathcal{B}_2$ denote the right ideal of $\mathcal{B}(X)$ whose elements are the operators in $\mathcal{B}(X)$ with range in $V^+$; and let $\mathcal{B}_3$ be the subalgebra of $\mathcal{B}(X)$ consisting of those operators with range in $\pi_2X_2$ and whose adjoints have range in $(\pi_2X_2)^+$. Then an application of Lemma 1 and Lemma 2 yields

$$\mathcal{B}(X) = \mathcal{A}_1 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_3.$$
Next we will show that $\mathfrak{A}$ is a right annihilator algebra. By the proof of Theorem 3.1 [2] we see that there exists an isomorphism $\alpha$ of $\pi_2 X_2$ onto $V^*$ such that $\alpha(\pi_2 x_2)(v) = v(x_2)$ for all $x_2 \in X_2$, $v \in V$.

We establish next an isomorphism $\beta$ of $\mathfrak{A}$ onto $(\mathcal{B}(V))^*$. For $A \in \mathfrak{A}$ define $\beta A = \alpha A \alpha^{-1}$. If we let $T \in \mathcal{B}(V)$ be defined by $Tv = \pi^{-1} A^* \pi_1 v$ for all $v \in V$, then

$$(T^* v^*)(v) = v^* (\pi^{-1} A^* \pi_1 v) = (A^* \pi_1 v)(\alpha^{-1} v^*) = (\alpha A \alpha^{-1} v^*)(v),$$

and it follows that $\beta A = T^*$ and hence $\beta A \in (\mathcal{B}(V))^*$.

The mapping $\beta$ is onto since if $T^* \in (\mathcal{B}(V))^*$ we can define $A \in \mathcal{B}(X)$ by $A = \alpha^{-1} T^* \alpha P$, where $P$ is the continuous projection of $X$ onto $\pi_2 X_2$, with null space $V^+$; clearly $A \subset \pi_2 X_2$, and $\beta A = T^* \alpha P \alpha^{-1} = T^*$. To show that Range $A^* \subset \pi_1 V$ we proceed as follows. Suppose first that $T$ is a one-dimensional operator, then $T = \nu \otimes \nu$, where $\nu$ is the canonical embedding of $V$ into $V^*$. We then have $A = \alpha^{-1} (\nu \otimes \nu) \alpha P$ and for any $x^* \in X^*$, $x \in X$ we obtain $A x^*(x) = x^* [\alpha^{-1} (\nu \otimes \nu) \alpha P x] = \nu(\alpha P x) x^* (\alpha^{-1} \nu^*) = (P x)(x^* (\alpha^{-1} \nu^*))$ using the definition of $\alpha$. Now $x = P x + (I - P) x$ and $(I - P) x \in V^+$, so $\pi_v(x) = x(v) = P x(v)$, and hence $A x^* = x^* (\alpha^{-1} \nu^*) \pi_v \in \pi_1 V$. This shows that Range $A^* \subset \pi_1 V$ in case $T$ is a one-dimensional operator. Similarly if $T$ is of finite rank we obtain Range $A^* \subset \pi_1 V$. Finally, if $T$ is arbitrary in $\mathcal{B}(V)$, there exists a sequence $T_n$ of operators of finite rank such that $T_n \to T$. Let $A_n = \alpha^{-1} T_n^* \alpha P$ and $A = \alpha^{-1} T^* \alpha P$. Since $T_n \to T$, since $\pi_1 V$ is a closed subspace of $X^*$ and since Range $A_n^* \subset \pi_1 V$ for each $n$, it follows that $A_n \to A$ and Range $A^* \subset \pi_1 V$. The mapping $\beta$ is therefore onto $(\mathcal{B}(V))^*$.

It is clear furthermore that $\beta$ is one to one, linear, bicontinuous and preserves multiplication, i.e., $\beta(AB) = (\beta A)(\beta B)$ for all $A, B \in \mathfrak{A}$. We can thus identify $\mathfrak{A}$ and $(\mathcal{B}(V))^*$. Now, $\mathcal{B}(V)$ is a left annihilator algebra [4, p. 107] so $(\mathcal{B}(V))^*$ is a right annihilator algebra. It follows that $\mathfrak{A}$ is a right annihilator algebra.

Next, if $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ then for any $x^* \in X^*$ and $x \in X$ we have $((AB)^* x^*) x = (A^* x^*) (B x) = 0$, since $A^* x^* \in \pi_1 V$ and $B x \in V^+$. Therefore $(AB)^* = 0$ so that $A B = 0$, hence $\mathfrak{B}$ annihilates $\mathfrak{A}$ on the right.

We consider next $\mathfrak{A} \oplus \mathfrak{B}$ which is the subalgebra consisting of those operators in $\mathcal{B}(X)$ with range in $\pi_2 X_2$ and whose adjoints have range in $\pi_1 X_1$. In order to show that $\mathfrak{A} \oplus \mathfrak{B}$ is a left annihilator algebra it suffices to notice that $\mathfrak{A} \oplus \mathfrak{B} = (\mathcal{B}(X_2))^*$. This equality follows from the fact that if $T \in \mathcal{B}(X_2)$ and $T = x_2 \otimes x_1$ with $x_2 \in X_2$ and $x_1 \in X_1 = X_1^*$.
then $T^{**} = \pi_{2X_3} \otimes \pi_{X_1} \in A_1 \oplus B_1$, so that $A_1 \oplus B_1$ and $(B(X_3))^{**}$ contain the same one-dimensional operators.

That $B_3$ annihilates $A_1 \oplus B_1$ on the left follows from the fact that if $A \in A_1 \oplus B_1$, $B \in B_3$, $x^* \in X^*$ and $x \in X$ we have $((BA)^* x^*) (x) = (B^* x^*) (Ax) = 0$, since $B^* x^* \in (\pi_2 X_2)^+$ and $Ax \in \pi_2 X_2$, so that $(BA)^* = 0$ and therefore $BA = 0$. The same conclusion holds if $A \in B_3$, $B_3$ is therefore nilpotent.

Finally we notice that if $A_1$ is a left annihilator algebra it is an annihilator algebra and so are $(B(V))^*$ and $B(V)$, consequently by [1] $V$ is a reflexive Banach space and so is $V^*$. But $V^*$ is isomorphic with $X_2$ by proof of Theorem 3.1 [2]; so $X_2$ is reflexive and by Lemma 3 we conclude that $X$ is reflexive. If $X$ is reflexive then $U = V^+ = (\pi_2 X_2)^+ = (0)$ which implies $B_i = (0)$ for $i = 1, 2$ and 3.

If $B_i = (0)$ this implies either $U = (0)$ or $V^+ = (0)$ or $(\pi_2 X_2)^+ = (0)$. But by Lemma 3 either one of these inequalities implies the other two so that $B_i = (0)$ and $A_1 \oplus B_1 = A_1$, and hence $A_1$ is a left annihilator algebra.

**Bibliography**


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