NOTE ON STAR-SHAPED SETS

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1. The aim of this note is to prove that if $M$ is a compact subset of $E^n$ and for some $m$ every $m$-dimensional hyperplane through a fixed point $p \in E^n$ intersects $M$ along a nonempty acyclic set, $1 \leq m \leq n-1$, then $M$ is star-shaped with respect to $p$, i.e., if $a \in M$ then the segment $\overline{pa}$ is contained in $M$.

This theorem is a generalization of a theorem of Aumann [1]. We gave recently a proof of Aumann's theorem based on the theory of multivalent mappings (see [3]); the present proof follows essentially the same line. The topological lemma on which it is based is susceptible to further generalizations; however, we give it here only in its simplest and easily proved case which is needed for the proof of the theorem about star-shaped sets.

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2. $H_n(X)$ will denote the $n$th Čech homology group of the space $X$ with the group $\mathbb{Z}_2$ of integers mod 2 as the group of coefficients. We will say that $X$ is acyclic if $X$ is connected and $H_n(X) = 0$, $n = 1, 2, 3, \ldots$.

Let $X$ be a compact metric space and let $\Phi: X \to 2^Y$ be an upper semi-continuous mapping of $X$ into the space $2^Y$ of all nonempty compact subsets of a space $F$. The triple $S = \{X, Y, \Phi\}$ will be called a family [2]. The set $X$ will be called the basis of $S$, the sets $\Phi(x)$—the elements of $S$, the set $\bigcup_{x \in X} \Phi(x) \subset Y$—the field of $S$. The field will be also denoted by $\Phi(X)$. A family $S$ is said to be acyclic if all its elements are acyclic.

$E^n$ will denote the Euclidean space, $D^n$ the unit $n$-ball in $E^n$ with center in the origin of coordinates $o$, $S^{n-1}$ will denote the boundary of $D^n$. $E = E^k$, $2 \leq k \leq n-1$, will stand for a fixed $k$-dimensional Euclidean subspace of $E^n$, $E' = E^{n-k}$ will be the orthogonal complement of $E$ in $E^n$.

For a fixed $r$, $1 \leq r \leq k-1$, $G_{k,r}$ will denote the grassmannian of (unoriented) $r$-planes in $E^k$. For every plane $x \subset E$ let $H(x)$ be the plane in $E^n$ spanned by $x$ and $E'$. If $x$ runs through $G_{k,r}$ then the correspondence $x \to H(x)$ is a one-to-one correspondence between $G_{k,r}$ and the set of all $(n-k+r)$-planes in $E^n$ containing $E'$.

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2.1. Lemma. Let \( \mathcal{F} = \{ G_k, E^n, \Phi \} \) be an acyclic family satisfying
\[
(H(x) \cap S^{n-1}) \subseteq \Phi(x)
\]
for every \( x \in G_k \). Then \( D^n \subseteq \Phi(G_k) \).

Proof. For \( r = 1 \) the lemma was proved in [2, 4a]. (The sentence
in parentheses on the bottom of p. 295 in [2] is incorrect; the proof
however is correct, provided \( m = 2 \).
To prove it in the general case we consider a fixed \((n-k+r-1)\)-plane \( E'' \) in \( E^n \) containing \( E' \). Let
\[
G' = \{ x \in G_k : H(x) \supset E'' \}
\]
Since the family \( \mathcal{F} \) restricted to \( G' \) is a family with \( G_{k-r+1,r} \) as basis,
we infer that \( \Phi(G') \supset D^n \). Since \( \Phi(G_k) \supset \Phi(G') \), this proves the lemma.

2.2. Lemma. Let \( \mathcal{F} = \{ G_k, E^n, \Phi \} \) be an acyclic family satisfying
\[
(E' \cap S^{n-1}) \subseteq \Phi(x) \subseteq H(x)
\]
for every \( x \in G_k \). Then \( (E' \cap D^n) \subseteq \Phi(G_k) \).

Proof. We will consider the grassmannian \( G_{k-k-r} \) of all \((k-r)\)-planes in \( E \). For every \( x \in G_{k-k-r} \), \( x^* \) will
denote the orthogonal complement of \( x \) in \( E \), and \( S(x) = x \cap S^{n-1} \). For any two sets \( A, B \subseteq E^n \) let \( A \ast B \) be
the union of all segments \( ab, a \in A, b \in B \). It is obvious that if \( B \subseteq H(x^*) \) then \( S(x) \ast B \) is homeomorphic to
the join of \( S(x) \) with \( B \). In particular, this implies
(i) If \( B \) is compact and acyclic and \( B \subseteq H(x^*) \) then \( S(x) \ast B \) is
also compact and acyclic.

Let \( h : E^n \to E^n \) be a homeomorphism of \( E^n \) onto itself satisfying the
following conditions
(ii) \( h(E' \cap D^n) \subseteq D^n \);
(iii) For every \( x \in G_{k-k-r} \),
\[
h(S(x) \ast (E' \cap S^{n-1})) = H(x) \cap S^{n-1}
\]
It is easy to construct such a homeomorphism.
Now, for every \( x \in G_{k-k-r} \) we put \( \Phi_1(x) = h(S(x) \ast \Phi(x^*)) \). It follows
from (i) that \( \mathcal{F}_1 = \{ G_{k-k-r}, E^n, \Phi_1 \} \) is an acyclic family. Moreover,
since \( \Phi(x^*) \supset E' \cap S^{n-1} \) we have \( S(x) \ast \Phi(x^*) \supset S(x) \ast (E' \cap S^{n-1}) \) and
(iii) implies \( \Phi_1(x) \supset H(x) \cap S^{n-1} \). Therefore the family \( \mathcal{F}_1 \) satisfies the
conditions of Lemma 2.1 and we infer that
(iv) \( D^n \subseteq \Phi_1(G_{k-k-r}) \).
Let \( y \in E' \cap D^n \). By (ii) and (iv) \( h(y) \in \Phi_1(x) \) for some \( x \in G_{k-k-r} \).
Therefore \( y \in h^{-1}(\Phi_1(x)) = S(x) \ast \Phi(x^*) \). Since \( S(x) \ast \Phi(x^*) \cap E' = \Phi(x^*) \cap E' \) it follows that \( y \in \Phi(x^*) \). Thus \( (E' \cap D^n) \subseteq \Phi(G_k) \)
which completes the proof.

2.3. Remark. Actually, a much stronger lemma holds. Namely, if \( \mathcal{F} = \{ G_k, E^n, \Phi \} \) is an acyclic family satisfying \( (E' \cap S^{n-1}) \subseteq \Phi(x) \),
then for some \( x \in G_k, \Phi(x) \cap H^*(x) \neq 0 \). This implies easily 2.2 and
may be proved using methods from [3].
3. **Theorem.** Let $M \subset E^n$ be a compact set, and $m$ a natural number, $1 \leq m \leq n - 1$. If there exists a point $p \in E^n$ such that for every $m$-plane $H$ through $p$, $H \cap M$ is acyclic then $M$ is star-shaped with respect to $p$.

**Proof.** We remark first that it follows from [3, 2.1] that $p \in M$.

Now let $a \in M$, $a \neq p$, and $L$ be the line through $a$ and $p$.

Suppose first that $m = 1$. Then $a, p \in L \cap M$ and $L \cap M$ is connected. Thus $ap \subset L \cap M$, which was to be proved.

Now let $2 \leq m \leq n - 1$. Let $S$ be the $(n - 1)$-sphere in $E^n$ such that $ap$ is its diameter, let $E$ be the $(n - 1)$-plane in $E^n$ orthogonal to $L$ and passing through the midpoint of $ap$.

For every $(m - 1)$-plane $x$ in $E$ we define $\Phi(x) = H(x) \cap M$, where $H(x)$ is as before the $m$-plane in $E^n$ spanned by $x$ and $L$. Thus $H(x)$ passes through $p$ and $\Phi(x)$ is acyclic. Therefore $\mathcal{F} = \{G_{a-1,m-1}, E^n, \Phi\}$ is an acyclic family. Obviously, $S \cap L = \{a, p\} \subset \Phi(x) \subset H(x)$. Hence $\mathcal{F}$ satisfies all conditions from Lemma 2.2 (with $k = n - 1$, $r = m - 1$) and it follows that $ap \subset \bigcup \Phi(x) \subset M$. This completes the proof of the theorem.

**References**


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