Thus every neighbourhood of x contains points y such that $y \in B$ and $P(y, A) = 0$. Hence

$$0 \leqq P(y, A_1) \leqq P(y, A) = 0.$$

But this contradicts the continuity of $P(x, A)$.

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DARBOUX FUNCTIONS OF BAIRE CLASS ONE AND DERIVATIVES

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Introduction. Let $I_0 = [0, 1]$ and let R be the reals. Let B_1 be the class of functions $f: I_0 \to R$ of Baire type at most one, and denote by D the class of functions $f: I_0 \to R$ which possess the Darboux property, i.e., take connected sets into connected sets. The class $B_1 \cap D$ is abbreviated by (B_1, D). If Δ is the class of functions $f: I_0 \to R$ which are derivatives, then we have the well-known relation $\Delta \subset (B_1, D)$. It is of interest to have characterizations for the classes Δ and (B_1, D). In this paper two characterizations of (B_1, D) are given as well as a characterization of Δ. This characterization together with one characterization of (B_1, D) provides a measurement by how much a function in (B_1, D) may fail to be in Δ.

Throughout the paper we will use the following notation. For $A \subset I_0$, A^0 is the interior of A relative to I_0, $\bar{A}$ stands for the closure of A, and $|A|$ denotes the Lebesgue measure of A.

First characterization of (B_1, D). We have occasion to use the following characterizations of B_1. (1) $f \in B_1$ if and only if for each $a \in R$ the sets $\{x : f(x) \geqq a\}$, $\{x : f(x) \leqq a\}$ are G_δ; (2) $f \in B_1$ if and only if every perfect subset P of I_0 has a point of continuity of $f|P$ (f restricted to P) [**3**].

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THEOREM 1. *$f\in(B_1, D)$ if and only if for each $a\in R$, the sets $\{x:f(x)\geqq a\}$, $\{x:f(x)\leqq a\}$ are G_δ with compact components.*

PROOF. Let $f\in(B_1, D)$, and let $a\in R$. By (1) we only need to show that the components of $\{x:f(x)\geqq a\}$, $\{x:f(x)\leqq a\}$ are compact. Let Q be a component of $E=\{x:f(x)\geqq a\}$. We may suppose that Q is a nondegenerate interval with endpoints $\alpha<\beta$. If $f(\alpha)<a$, then the set $f([\alpha, \beta))$ would be disconnected, contradicting $f\in D$. Therefore, $\alpha\in E$ and similarly $\beta\in E$. Consequently, $Q=[\alpha, \beta]$, and the proof of the necessity is complete.

For the sufficiency we observe first that $f\in B_1$ by (1). If we deny $f\in D$, we have an interval $[\alpha, \beta]\subset I_0$ such that $f([\alpha, \beta])$ is not connected. For notational simplification we may assume that $[\alpha, \beta]=I_0$. There exists $a\in R$ such that the sets $E_1=\{x:f(x)\leqq a\}$, $E_2=\{x:f(x)\geqq a\}$ satisfy $I_0=E_1\cup E_2$, $E_1\neq\emptyset$, $E_2\neq\emptyset$, and $E_1\cap E_2=\emptyset$. Let $\{Q\}_1$ be the collection of components of E_1 which are nondegenerate, and let $\{Q\}_2$ be defined analogously. Let $\{Q\}=\{Q\}_1\cup\{Q\}_2$. By hypothesis, each $Q\in\{Q\}$ is compact, and two distinct components in $\{Q\}$ are disjoint. Hence the set $P=I_0-\bigcup Q^0$ is perfect, where the union is extended over all $Q\in\{Q\}$. By (2) there is a point $x_0\in P$ at which $f|P$ is continuous. Since $I_0=E_1\cup E_2$, we may assume that $x_0\in E_1$, and since $E_1\cap E_2=\emptyset$, we infer that $f(x_0)<a$. Therefore, there is a $\delta>0$ such that $I_\delta\cap P\subset E_1$, where $I_\delta=[x_0-\delta, x_0+\delta]\cap I_0$.

Let $Q\in\{Q\}_2$. We will prove that $I_\delta\cap Q=\emptyset$. For, if $I_\delta\cap Q\neq\emptyset$, there is an endpoint γ of Q which is in I_δ. Since $\gamma\in E_2\cap P$, this contradicts $I_\delta\cap P\subset E_1$. Since every $x\in E_2-\bigcup Q^0$ is in P, where the union is extended over all $Q\in\{Q\}_2$, we infer that $I_\delta\cap E_2=\emptyset$, and hence $I_\delta\subset E_1$. There is then $Q\in\{Q\}_1$ such that $I_\delta\subset Q$, and thus $x_0\in Q^0$, contradicting $x_0\in P$. This completes the proof.

REMARK. In the necessity of Theorem 1 we have actually proved that for every $f\in D$ the sets $\{x:f(x)\geqq a\}$, $\{x:f(x)\leqq a\}$ have compact components. Without the condition that these sets be G_δ the converse is not true as the example $f(x)=1$, x rational, $f(x)=0$, x irrational, shows.

APPLICATIONS. Theorem 1 can be applied to show that various derivatives are in (B_1, D). As an application, we will consider approximate derivatives and nth Peano derivatives. For the properties of approximate derivatives that will be used see [2], and for those of the nth Peano derivative see [4]. In these papers it has been shown that approximate derivatives and nth Peano derivatives are in (B_1, D). We will show that Theorem 1 provides a very simple proof of this fact.

THEOREM 2. *If* $f: I_0 \to R$ *is an approximate derivative or nth Peano derivative, then* $f \in (B_1, D)$.

PROOF. It is known that $f \in B_1$ [2; 4]. If Q is a component of $\{x: f(x) \geqq a\}$, simple arguments used in the above mentioned papers show that $\overline{Q} \subset \{x: f(x) \geqq a\}$, and thus the components of $\{x: f(x) \geqq a\}$ are compact.

Theorem 1 can be applied to provide a simple proof of a characterization of (B_1, D) due to Zahorski [5]. Let $f: I_0 \to R$, and let for $a \in R$, $E_a = \{x: f(x) > a\}$, $B_a = \{x: f(x) < a\}$.

DEFINITION (ZAHORSKI). $f \in M_0 (f \in M_1)$ if and only if for each $a \in R$ the sets E_a, B_a are F_σ and each point of E_a, B_a is a bilateral point of accumulation (condensation) of E_a, B_a, respectively.

THEOREM 3. $M_0 = M_1 = (B_1, D)$.

PROOF. The inclusion $M_1 \subset M_0$ is obvious, and $(B_1, D) \subset M_1$ is easy to prove. It suffices to show that $M_0 \subset (B_1, D)$. Since $f \in M_0$ implies that $f \in B_1$, for each $a \in R$, the sets $\{x: f(x) \geqq a\}$, $\{x: f(x) \leqq a\}$ are G_δ. Let Q be a component of $E = \{x: f(x) \geqq a\}$. We may assume that Q is an interval with endpoints $\alpha < \beta$. If $f(\alpha) < a$, the point α would not be a right point of accumulation of $\{x: f(x) < a\}$. Hence $f(\alpha) \geqq a$, and similarly, $f(\beta) \geqq a$. Thus $Q = \overline{Q}$, and by Theorem 1 the proof is complete.

Second characterization of (B_1, D). Let $\{I\}$ be the collection of all nondegenerate compact subintervals of I_0. We shall use the notation $I \to x$ to denote $x \in I$ and $|I| \to 0$. Following Gleyzal [1] we say that an interval function $\phi: \{I\} \to R$ *converges* on I_0 if and only if for each $x \in I_0$, $\lim_{I \to x} \phi(I)$ exists. In [1], Gleyzal proved the following characterization of B_1: $f \in B_1$ if and only if f is the limit of a convergent interval function.

DEFINITION. A function $f: I_0 \to R$ is said to possess *property* (C_1) if and only if for each $I \in \{I\}$ there exists $x_I \in I^0$ such that $I \to x$ implies that $f(x_I) \to f(x)$, for each $x \in I_0$.

REMARK. A comparison with ordinary continuity may clarify the condition (C_1). A function $f: I_0 \to R$ is continuous on I_0 if and only if (*) $I \to x$, $x_I \in I^0$ implies $f(x_I) \to f(x)$, for each $x \in I$. Condition (C_1) is a formulation of the *weakest* version of (*) in the sense that we only require that in each I^0 there is a point x_I so that $I \to x$ implies $f(x_I) \to f(x)$.

THEOREM 4. $f \in (B_1, D)$ *if and only if f has property* (C_1).

PROOF. First, suppose that f has property (C_1). If $\phi(I)=f(x_I)$, then $\phi(I)\to f(x)$ as $I\to x$. Consequently, by Gleyzal's theorem, $f\in B_1$. In order to show that $f\in D$, we only need to show, by Theorem 1, that the sets $\{x:f(x)\geqq a\}$, $\{x:f(x)\leqq a\}$ have compact components. Let Q be a component of $E=\{x:f(x)\geqq a\}$. We may assume that Q is a nondegenerate interval with endpoints $\alpha<\beta$. Let $x_n\in(\alpha,\beta)$ with $x_n\to\alpha$, and let $I_n=[\alpha, x_n]$. Then $I_n\to\alpha$, and thus $f(x_{I_n})\to f(\alpha)$. Since $x_{I_n}\in I_n^0\subset Q$, we have $f(x_{I_n})\geqq a$, and thus $f(\alpha)\geqq a$. Similarly, $f(\beta)\geqq a$, and hence $Q=\overline{Q}$.

The necessity is more complicated. We assume that $f\in(B_1, D)$. Since $f\in B_1$, there is a sequence of continuous functions $f_n: I_0\to R$ such that $f_n\to f$ on I_0. For each n, select a finite number of intervals $I_{n1},\cdots, I_{nj_n}$ in $\{I\}$ such that $I_0=I_{n1}^0\cup\cdots\cup I_{nj_n}^0$, $|I_{nk}|<1/n$, $k=1,\cdots, j_n$, and the saltus of f_n on each I_{nk} is $<1/n$. Define $\{I\}_1=\{I_{nk}, k=1,\cdots, j_n; n=1,2,\cdots\}$. We note that $\{I\}_1$ covers I_0 in the sense of Vitali.

Let $\{I\}_2$ be the collection of all $I\in\{I\}$ for which there exists $J\in\{I\}_1$ such that $I\subset J$, and let $\{I\}_3$ be the collection of the remaining intervals. For $I\in\{I\}_2$, let n_0 be the largest integer for which there exists k, $1\leqq k\leqq j_{n_0}$, such that $I\subset I_{n_0k}$. Among the k's for which $I\subset I_{n_0k}$ select one k_0, and call (n_0, k_0) the *pair associated with I*.

Let $I\in\{I\}_2$ and let (n,k) be the pair associated with I. Since $f\in D$, $\overline{f(I)}$ is a closed interval, and since f_n is continuous, $f_n(I_{nk})$ is a compact interval. Hence there are points $y_{nk}\in I_{nk}$, $\xi_I\in\overline{f(I)}$ such that

$$|f_n(y_{nk})-\xi_I| = \operatorname{dist}[f_n(I_{nk}), f(I)].$$

Since $f\in D$, we can choose $x_I\in I^0$ such that $|\xi_I-f(x_I)|<1/n$. If $I\in\{I\}_3$, select any $x_I\in I^0$. For each $I\in\{I\}$ we have now chosen a point $x_I\in I^0$. We will verify that $f(x_I)\to f(x)$ if $I\to x$. We will proceed in several steps.

STEP 1. Let $I\in\{I\}_2$ and let (n,k) be the pair associated with I. Then $I\subset I_{nk}$. We define $\phi_1(I)=f_n(y_{nk})$. We will show that ϕ_1 converges to f on I_0. This part of the proof is essentially a reproduction of Gleyzal's argument [1] with some modifications. Let $\xi\in I_0$ and let $\{I_j\}\subset\{I\}_2$ be a sequence with $I_j\to\xi$. For each n, there is a $j_n>0$ such that for $j>j_n$, $I_j\subset I_{nk}$ for some k. Hence, for $j>j_n$, $|I_{n_jk_j}|<1/n$, where (n_j,k_j) is the pair associated with I_j. Since $n_j\geqq n$, we have that $n_j\to\infty$ and $I_{n_jk_j}\to\xi$ as $j\to\infty$.

Let p be a positive integer. There is $l>0$ such that $|f_n(\xi)-f(\xi)|<1/p$ for $n>l$. There is $l'>0$ such that $j>l'$ implies that $n_j>\max(p,l)$.

Let $j > \max(l', l)$. Then

$$\begin{aligned} |\phi_1(I_j) - f(\xi)| &= |f_{n_j}(y_{n_jk_j}) - f(\xi)| \\ &\leqq |f_{n_j}(y_{n_jk_j}) - f_{n_j}(\xi)| + |f_{n_j}(\xi) - f(\xi)| \\ &< \frac{1}{n_j} + \frac{1}{p} < \frac{2}{p}\cdot \end{aligned}$$

Hence $\phi_1(I_j) \to f(\xi)$.

STEP 2. For $I \in \{I\}$, let $\phi(I) = f(x_I)$, where $x_I \in I^0$ is the point chosen above. We will prove that ϕ converges to f on I_0. Let $\xi \in I_0$ and let $\{I_j\} \subset \{I\}$ be a sequence with $I_j \to \xi$. There is $j_1 > 0$ such that $j > j_1$ implies $I_j \subset I_{1k}$ for some k. Hence for $j > j_1$, $I_j \in \{I\}_2$. Consequently, by step 1, $\phi_1(I_j) \to f(\xi)$. For $j > j_1$, let (n_j, k_j) be the pair associated with I_j. Then $I_j \subset I_{n_jk_j}$, $I_{n_jk_j} \to \xi$, and $\phi_1(I_j) = f_{n_j}(y_{n_jk_j})$. Thus

$$\begin{aligned} |\phi(I_j) - \phi_1(I_j)| &= |f(x_{I_j}) - f_{n_j}(y_{n_jk_j})| \\ &\leqq |f(x_{I_j}) - \xi_{I_j}| + |\xi_{I_j} - f_{n_j}(y_{n_jk_j})|, \end{aligned}$$

where ξ_{I_j} has been defined previously. Thus

$$|\phi(I_j) - \phi_1(I_j)| < \frac{1}{n_j} + \operatorname{dist}[f(I_j), f_{n_j}(I_{n_jk_j})] \leqq \frac{1}{n_j} + |f_{n_j}(\xi) - f(\xi)|.$$

Since $f_{n_j}(\xi) \to f(\xi)$ as $j \to \infty$, we infer that $|\phi(I_j) - \phi_1(I_j)| \to 0$. Hence $\phi(I_j) \to f(\xi)$, completing the proof.

Characterization of Δ. As before, let $\{I\}$ be the collection of all nondegenerate compact intervals of I_0.

DEFINITION. A function $f: I_0 \to R$ is said to have property (C_2) if and only if for each $I \in \{I\}$ there is $x_I \in I^0$ such that

(1) $I \to x$ implies that $f(x_I) \to f(x)$, for each $x \in I_0$,

(2) if $I = I_1 \cup I_2$, $I_1^0 \cap I_2^0 = \varnothing$, then

$$f(x_I) = \frac{f(x_{I_1})|I_1| + f(x_{I_2})|I_2|}{|I_1| + |I_2|}\cdot$$

REMARK. Condition (1) is the same as (C_1). Condition (2) says that $f(x_I)$ is between $f(x_{I_1})$ and $f(x_{I_2})$. One may refer to (2) by saying that $f(x_I)$ is *mean-valued*. We note that (2) is the same as saying that $f(x_I)|I|$ is an additive interval function.

THEOREM 5. *$f \in \Delta$ if and only if f has property (C_2).*

PROOF. Suppose $f \in \Delta$. Then there exists $F: I_0 \to R$ such that on I_0, $F'(x) = f(x)$. Let $I = [a, b]$ be in $\{I\}$. Then there exists $x_I \in I^0$

such that $F(a)-F(b)=f(x_I)|I|$. Thus the interval function $f(x_I)|I|$ is additive. If $I_j \to x$, $I_j=[a_j, b_j]$, then

$$f(x_{I_j}) = \frac{F(b_j) - F(a_j)}{b_j - a_j} \to f(x).$$

Conversely, assume that f has property (C_2). Then, for each $I \in \{I\}$ there is $x_I \in I^0$ with the properties (1) and (2). Define $F(0)=0$ and for $0<x\leqq 1$, define $F(x)=f(x_I)|I|$, where $I=[0, x]$. Since $f(x_I)|I|$ is an additive interval function, it follows that for $I=[a, b]$,

$$F(b) - F(a) = f(x_I)(b - a).$$

Let $x \in I_0$ and let $\{I_j\} \subset \{I\}$ be a sequence with $I_j \to x$, and set $I_j=[a_j, b_j]$. Then, since $f(x_{I_j}) \to f(x)$, we infer that

$$\frac{F(b_j) - F(a_j)}{b_j - a_j} \to f(x).$$

Thus $F'(x)=f(x)$, and the proof is complete.

REMARK. Comparing Theorem 5 with Theorem 4, we observe that the condition (2) in (C_2) is precisely the property by which a Darboux function of Baire class one may fail to be a derivative.

It is possible to rewrite Theorem 5 in a slightly different form. An interval function ϕ: $\{I\} \to R$ will be termed *mean-valued* if and only if $\phi(I)|I|$ is additive.

THEOREM 6. *$f \in \Delta$ if and only if f is the limit of a convergent and mean-valued interval function.*

The proof is the same as the one above.

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